

# Estimating a Signal In the Presence of an Unknown Background

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## Abstract

We describe a method for fitting distributions to data which only requires knowledge of the parametric form of either the signal or the background but not both. The unknown distribution is fit using a non-parametric kernel density estimator. A transformation is used to avoid a problem at the data boundaries. The method returns parameter estimates as well as errors on those estimates. Simulation studies show that these estimates are unbiased and that the errors are correct.

*Keywords:* Maximum likelihood, coverage, Monte Carlo, transformations, bootstrap

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## 1. Introduction

Almost all data sets encountered in High Energy Physics (HEP) have events that were generated by different mechanisms. Interest often focuses on one of these mechanisms, what often is called a signal. By judicious cutting on auxiliary variables it is often possible to bring out the signal, that is, to improve the signal to noise ratio, but it is usually not possible to eliminate the noise, or background events, altogether. In many cases the researchers want to estimate the parameters involved in the signal such as the location and the width of an invariant mass peak, together with their statistical errors. In order to do so, they need to find a model for the data, and because the data still contains background events the model usually has the form of a mixture distribution. In practice it often turns out to be fairly straightforward to find a parametric description for the signal density, for example from theoretical considerations. In contrast it is often very difficult to model the background density, and the typical approach is one of trial and error: fit a number of shapes until a satisfactory one is found. This approach has several drawbacks: (i) it is quite time consuming, (ii) different researchers might end up with different parametrizations, (iii) it is hard to know when to stop in order not to overfit the data and (iv) it is almost impossible to gauge the systematic uncertainty of an incorrectly specified background on the parameter estimates.

There are, however, other solutions to the problem of estimating a density, namely the so-called nonparametric density (NPD) estimators. A number of such methods are known, such as kernel methods, nearest neighbor methods, the method of penalized likelihood and others. We could apply any of these methods to HEP data but unfortunately they do not yield the parameters that are the ultimate goal of the fitting process.

The semiparametric method that will be discussed here combines traditional parametric fitting with nonparametric density estimation. A parametric description is used for the signal, the background is modeled nonparametrically and these two ingredients are combined into one fitting function.

This method is made especially useful by two features of HEP data. First, we often have available a sample of pure background events, either by eliminating potential signal events by cutting on auxiliary variables or from Monte Carlo. This will allow us to employ the most powerful techniques developed for NPD estimation, for example the methods for choosing the optimal amount of smoothing. Secondly, data in HEP is usually supported on a finite interval. This normally would lead to the so-called boundary problem in NPD density estimation, a seemingly innocuous problem for which to date no single solution is known. In this work we solve this problem by transforming the data, a solution which is viable because in HEP we are not interested in the NPD estimate itself but in the signal parameters. As our simulation studies show this solution of the boundary problem can be applied in a wide variety of situations and leads to correct estimates of the parameters and their errors.

There is a large volume of literature in the field of Statistics on nonparametric density estimation. Standard references include Scott [1], Silverman [2], Tapia

and Thompson [3], Fryer [4] and Bean [5]. Kernel density estimation was first introduced independently by Rosenblatt [6] and Parzen [7].

One of the early discussions of NPD estimation in HEP was by Cranmer [8] in 2001, who also discussed their use in multivariate analysis [9]. It has since been used in a variety of HEP analyses ([10], [11], [12], [13]).

## 2. Kernel Density Estimation

Nonparametric density estimation is concerned with estimating a probability density without assuming a functional form for the density. The “nonparametric” in the name is slightly misleading because these estimators have in fact something like parameters. There are a number of different methods that have been studied in some detail, such as nearest neighbor estimators, splines, or orthogonal series, and so on. Even a histogram (properly scaled) can be viewed as a NPD estimator. We will concentrate on the best known and most widely used method, the kernel density (KD) estimator.

Let  $X_1, \dots, X_n$  be observations from some unknown density  $f$ . Let  $K$  be any continuous, nonnegative and symmetric function with  $K(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . Often  $K$  is chosen to be a probability density itself, say a Gaussian density. We then get the following definition of a nonparametric kernel density estimator:

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) \quad (1)$$

As has been noted many times, the choice of the kernel function is of minor importance (Scott [1]). We will use the Gaussian density in this work but other kernels could also be chosen.

The  $h$  is a tuning parameter called the bandwidth. It governs the variance-bias trade-off, with a small value of  $h$  resulting in a very ragged curve whereas a large value oversmoothes the density.

How do we choose the bandwidth  $h$ ? This has been one of the most active areas of research in the field of Statistics in the last 20 years, and a number of methods have been proposed. To begin with a criterion for “best” is needed, and most research has focused on either the *Integrated Squared Error*

$$ISE(h) = \int_{-\infty}^{\infty} (\hat{f}_h(x) - f(x))^2 dx \quad (2)$$

or the *Mean Integrated Squared Error*  $MISE(h) = E[ISE(h)]$ , the expected value of  $ISE(h)$ . Clearly both depend on the unknown density  $f$  and therefore have to be estimated from the data. Another criterion is the *Asymptotic Mean Integrated Squared Error*  $AMISE(h)$ , based on a consideration of the large sample behavior of the  $MISE$ . Alternatively one can consider the corresponding *Mean Integrated Absolute Value Error* criterion (Devroye and Györfi [14]). Unfortunately, there is no obvious way to choose among these criteria, although Jones [15] gives some strong arguments in favor of  $MISE(h)$ . Detailed

discussions on bandwidth selection can be found in Turlach [16] and Chiu [17]. In this paper we will not advocate the use of one specific criterion but rather we will vary  $h$  over a range of reasonable values, thereby gaining insight into the systematic uncertainty introduced by this choice.

It is intuitively obvious, and can be shown to be mathematically correct, that, in regions where the true density changes slowly, the optimal value of  $h$  should be larger than in regions where  $f$  changes rapidly. It is therefore reasonable to use an adaptive bandwidth as follows:

$$\hat{f}_{h_x}(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h_x}\right) \quad (3)$$

One suggestion is to use a fixed bandwidth  $h$  to get a pilot estimate  $\hat{f}_h$  and then recompute the density estimate using  $h_x = h/\sqrt{\hat{f}_h(x)}$  (Abramson [18]). In our simulation studies there was very little difference between the adaptive method and using a fixed bandwidth, but it is a good idea to investigate this issue in any specific situation.

For the simulation studies shown later we used a simple bandwidth formula suggested by Scott [1]: let  $s$  be the standard deviation of the data and let  $IQR$  be the interquartile range, then

$$h = 1.06 \min\{s, IQR/1.34\} n^{-1/5} \quad (4)$$

The extension to multivariate data is straightforward and usually done using a product kernel:

$$\hat{f}_{h_1, \dots, h_d}(x_1, \dots, x_d) = \frac{1}{nh_1 \dots h_d} \sum_{i=1}^n \prod_{j=1}^d K_j\left(\frac{x_j - X_j^i}{h_j}\right) \quad (5)$$

### 3. The Problem of Boundaries

One typical feature of the data sets in HEP is that they are truncated, i.e., the data are bounded. For univariate data the boundary consists simply of the two endpoints of an interval  $[A, B]$ . Unfortunately nonparametric density estimators have a difficult time dealing with boundaries. The reason for this is that the lack of data beyond the boundary leads to a smaller estimate just inside the boundary and, because they have to integrate to 1, they overestimate the density in the middle.

A number of methods have been proposed to deal with the boundary problem. One approach is to modify the kernel near the boundary (Gasser, Müller and Mammitzsch [19], Jones [20]). The most serious problem with these modified boundary kernel methods is that they often lead to negative density estimates.

Another idea is to fit local polynomials of low order as discussed in Cheng, Fan and Marron [21]. The argument is that local polynomial estimation would

automatically correct for boundary effects in regression, and should therefore also work in density estimation (Fan and Gijbels [22]). A boundary correction, though, takes place only if the polynomial is of the “correct” order, else it can make the boundary effect even larger. Local polynomial fitting has not been as successful in density estimation as it is in regression problems, although Zhang and Karunamuni [23] improved the performance of this method by combining it with a bandwidth variation function.

Another set of boundary correction methods modifies the bandwidth near the boundaries. The basic idea is that a larger bandwidth close to the boundary should compensate for the lack of data beyond it. Specific proposals can be found in Rice [24], Gasser [19] and Müller [25]. In contrast Dai and Sperlich [26] actually advocate a smaller bandwidth.

One of the early attempts at solving the boundary problem was the reflection method, introduced by Schuster [27] and Silverman [2] and later extended by Cline and Hart [28]. A more recent extension of this method is to create pseudo data to correct for edges (Cowling and Hall [29].) This method is more adaptive than the common data reflection approach in the sense that it corrects also for discontinuities in derivatives of the density. There is also a whole class of transformation methods (Wand, Marron and Ruppert [31], Ruppert and Marron [32], and Yang [33]). In addition, Zhang, Karunamuni and Jones [30] suggested a method of generating pseudo data combining the transformation and reflection methods.

Unfortunately, despite the numerous methods that have been proposed, none has been shown to yield acceptable results in a wide range of cases. In this paper we will take advantage of the fact that our ultimate goal is not density estimation but the estimation of parameters such as the number of signal events. We can therefore transform the data to the whole real line and simply do the fitting there. Note that these are transformations of the data that leave the parameters unchanged.

Say the original observations are  $X_1, \dots, X_n$  and they are located in the interval  $[A, B]$ . Then we will calculate new observations  $X_1^T, \dots, X_n^T$  with  $X_k^T = H(X_k)$ ;  $k = 1, \dots, n$ . The transformation  $H$  can be any function that is monotonically increasing, continuous and that maps  $[A, B] \rightarrow (-\infty, \infty)$ . We will use  $H(x) = \log\left(\frac{x-A}{B-x}\right)$ .

When we transform the data, what happens to the density of the original data and to its parameters? Say the original density was given by  $f(x; \theta)$  where  $\theta$  is the vector of parameters. Let’s denote the density of the transformed data by  $g$ . Then

$$g(y; \theta) = f(H^{-1}(y); \theta) \cdot \frac{d}{dy}H^{-1}(y) \quad (6)$$

For the transformation above we have

$$H^{-1}(y) = \frac{A + Be^y}{1 + e^y} \text{ and } \frac{d}{dy}H^{-1}(y) = \frac{(B - A)e^y}{(1 + e^y)^2} \quad (7)$$

and (6) becomes

$$g(y; \theta) = f\left(\frac{A + Be^y}{1 + e^y}; \theta\right) \cdot \frac{(B - A)e^y}{(1 + e^y)^2} \quad (8)$$

We can now find the parameter estimates  $\hat{\theta}$  of  $\theta$  via maximum likelihood. The log-likelihood function  $l(\theta)$  is given by

$$l(\theta) = \sum_{i=1}^n \log g(X_i^T; \theta) = \sum_{i=1}^n \log f(X_i; \theta) \cdot \frac{(B - A)e^{X_i^T}}{(1 + e^{X_i^T})^2} \quad (9)$$

and we can see that it has a rather simple form.

One small problem with the transformation approach is that it makes visualization of the fit somewhat difficult. Of course we would like to visualize the fit on the original data. To achieve this we need to “back-transform” the estimated density. If  $\hat{g}(y)$  is the density estimate of the transformed data at the point  $y$ , then the corresponding estimate of the original data at the point  $x = H^{-1}(y)$  is given by

$$\hat{f}(x) = \hat{g}(H(x)) \cdot \frac{d}{dx} H(x) = \hat{g}(H(x)) \cdot \frac{B - A}{(x - A)(B - x)} \quad (10)$$

and we see that this function has singularities at the boundary points  $A$  and  $B$ , even though it still is a proper density which integrates out to 1. Because the fitting is done entirely in the transformed space this does not affect the parameter estimates.

An added benefit to this approach is that the transformed data becomes more “Gaussian-like” and as a consequence the formula for the bandwidth  $h$  (Eq. 4) is known to be close to optimal. The use of transformations for this purpose was discussed in Hartert [34].

#### 4. Semiparametric Fitting

The idea of mixing parametric and nonparametric components in a model is quite common in regression problems but has received fairly little attention in density estimation. For some work in the statistics literature see Olkin and Spiegelman [39], Schuster and Yakowitz [40] and Faraway [41].

In HEP numerous papers have used the idea of estimating the background non-parametrically, though in general without a consideration of the boundary problem. Cranmer [42] presents a general discussion of KDE and suggests using the boundary kernel method or the reflection method in case of a boundary. Adelman [43] and Meyer [44] are two examples of the implementation of the boundary kernel method in a HEP analysis. Our solution to the boundary problem, though, is more generally applicable and easier to use.

Say we have a set of observations (or events)  $X_1, \dots, X_n$  with  $X_i \in [A, B]$ . We assume that each event was either generated by a background distribution with density  $f_B(x)$ , or by a signal distribution with density  $f_S(x; \theta)$ . Furthermore, we

assume that  $(1 - \alpha)100\%$  of the events come from the background distribution. Finally, we also have a sample of pure background events  $Y_1, \dots, Y_m$ .

Notice that the background distribution might also depend on some parameters, but because we will not use the functional form of  $f_B$  we do not need to worry about them. Then the density of the mixing distribution is given by

$$f(x; \alpha, \theta) = (1 - \alpha)f_B(x) + \alpha f_S(x; \theta) \quad (11)$$

We can use the maximum likelihood method to find the estimates  $\hat{\alpha}$  and  $\hat{\theta}$  of  $\alpha$  and  $\theta$  by maximizing the log-likelihood  $l(\alpha, \theta)$  given by

$$l(\alpha, \theta) = \sum_{i=1}^n \log f(X_i; \alpha, \theta) \quad (12)$$

In our situation we don't really know the functional form of  $f_B$ , and we would therefore like to replace  $f_B$  by the nonparametric kernel density estimator. To do this we first have to transform the data to get new observations  $X_1^T, \dots, X_n^T$  and  $Y_1^T, \dots, Y_m^T$  as described above. Next we use the background sample to find the optimal bandwidth  $h_{opt}$  and then we estimate the background density  $\hat{g}_B$  at the points  $X_1^T, \dots, X_n^T$  using  $h_{opt}$ .

The log-likelihood function then becomes

$$l(\alpha, \theta) = \sum_{i=1}^n \log \left\{ (1 - \alpha)\hat{g}_B(X_i^T) + \alpha f_S(X_i; \theta) \cdot \frac{(B - A)e^{X_i^T}}{(1 + e^{X_i^T})^2} \right\} \quad (13)$$

The maximization could now be done using MINUIT [35] or any other optimization routine.

If we want to do a hypothesis test for the presence of a signal  $H_0 : \alpha = 0$  vs  $H_a : \alpha > 0$  we can use the likelihood ratio test with test statistic

$$T = 2 \left( l(\hat{\alpha}, \hat{\theta}) - l(0, 0) \right) \quad (14)$$

Of course, the conditions of Wilks' theorem [36] are not satisfied here. Nevertheless, quite often  $T$  will have an approximate chi-square distribution with 1 degree of freedom anyway. Also, because it is a large sample theorem, even when it does apply one needs to check whether the asymptotic regime has been reached. This issue of course applies to the parametric method just as much as to the semiparametric one.

If instead we want to find a confidence interval or an upper bound for the parameters, we can find error estimates from the usual large sample theory of maximum likelihood estimators (Casella and Berger [38]). These worked quite well in the simulation studies we conducted. As an alternative one could find the error estimates via the statistical bootstrap (Efron [37]). An example of the use of kernel density estimation as well as the bootstrap method in Astrophysics is discussed in Adriani [45].

## 5. Case Study

As an illustration, consider the following: The background is itself a 50-50 mixture of a Beta distribution with shape parameters (1, 10) and a uniform distribution. This density covers two extreme cases: on the left boundary the true density has a large slope and on the right it has slope 0. The signal distribution is a Gaussian  $\mu = 0.35$  and  $\sigma = 0.02$ .

Figure 1 shows one histogram of 1000 pure background events and another of 1000 data events of which 50 are signal events. The other two histograms correspond to these same two samples but with the variable transformed. For the transformed histograms, their respective semiparametric fits are shown as continuous curves. Here we fit for both the number of signal events and the signal location. The estimates are  $51.0 \pm 9.8$  and  $0.354 \pm 0.005$ . (The estimates using the exact parametrization would have been  $52.9 \pm 9.8$  and  $0.353 \pm 0.005$ ). If instead we had used the bootstrap to estimate errors we would have found  $51.0 \pm 10.1$  and  $0.354 \pm 0.005$ . Figure 2 shows the results of a sensitivity study. Using the statistical bootstrap and 5 different bandwidth selection methods we find that the true optimal bandwidth is between 0.42 and 0.52. Varying  $h$  over this range we see that neither the estimates nor their errors depend on the exact choice of bandwidth  $h$ .

## 6. Simulation Study

We will consider six different cases:

**Case 1** background is a 80-20 mixture of a Beta(3,20) and a Beta(3,6). In this case the density goes to 0 as  $x$  goes to 0 or 1, so no transformation is necessary.

**Case 2** background is an exponential distribution with rate 1, truncated at  $x=1$ . An example of a density with a moderate slope on one side ( $x=0$ )

**Case 3** background is a 50-50 mixture of a Beta(1,10) and a uniform [0,1]. An example of a density with large slope on one end and slope 0 on the other.

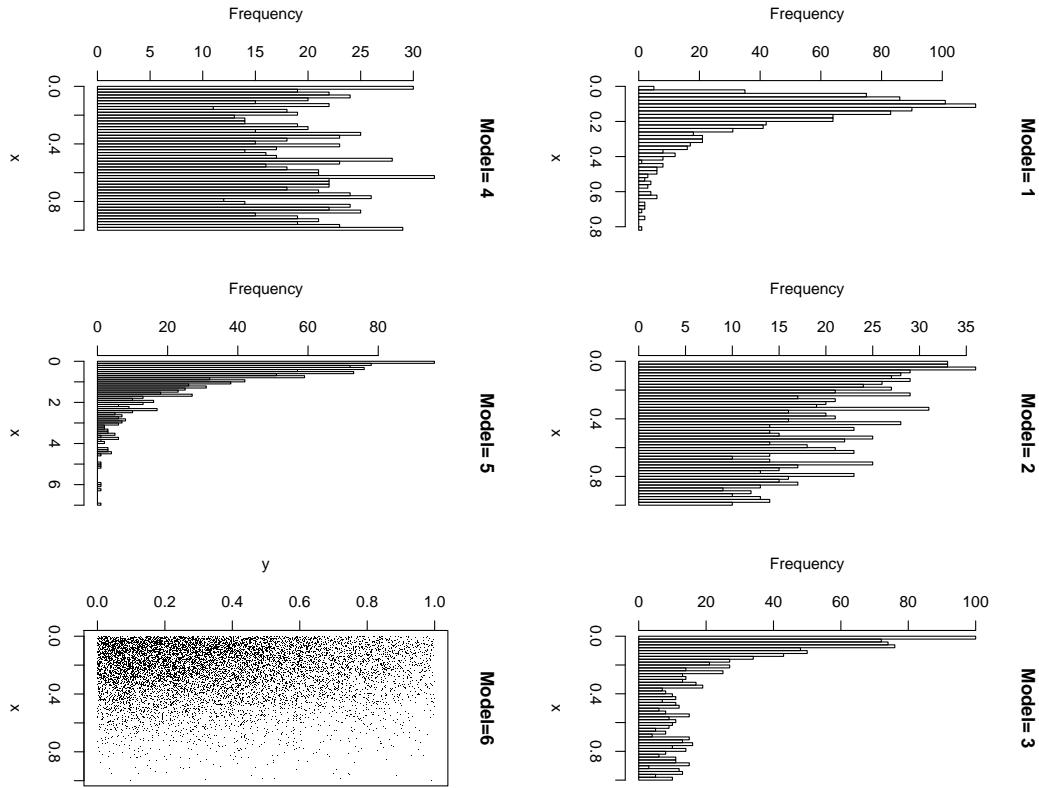
**Case 4** background is a uniform on [0,1]

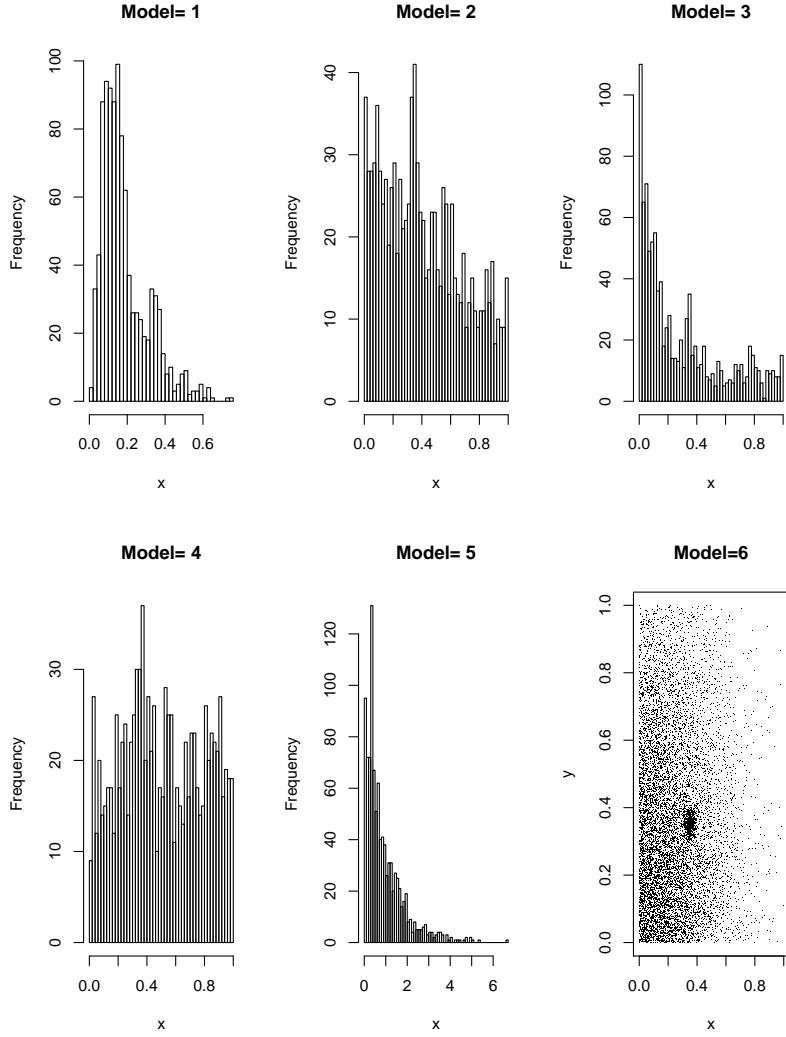
**Case 5** background is an exponential rate 1. An example with a boundary on just one side. For this we use the transformation  $Y = \log(X)$ .

**Case 6** an example of a multivariate problem. The background is a bivariate normal with means (0,0), standard deviations (0.3, 0.5) correlation 0 and truncated to  $[0, 1]^2$ .

In cases 1-5 a signal is modeled as a Gaussian mean 0.35 and standard deviation 0.02. In case 6 it is a bivariate Gaussian with means (0.35, 0.35) standard deviations (0.02, 0.02) and correlation 0.

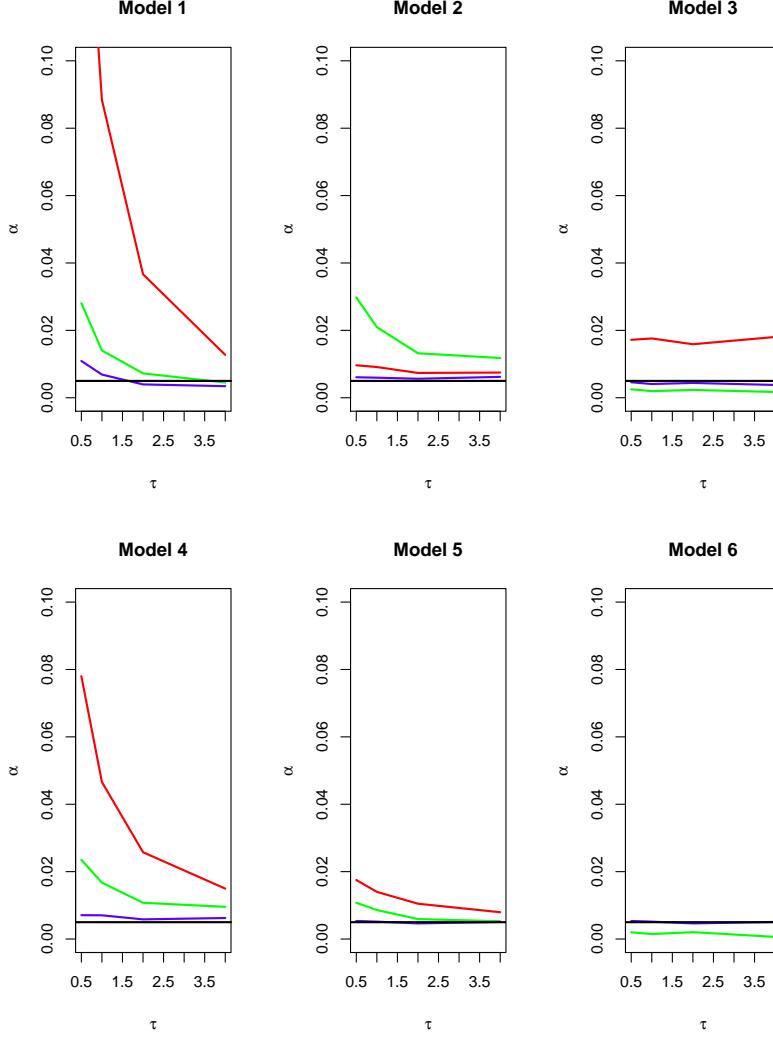
Examples are shown in figures 1 (1000 background events) and figure 2 (950 background events and 50 signal events).





We begin with the test for the presence of a signal. Here for the method to work it has to achieve the nominal type I error probability  $\alpha$ . We will use  $\alpha = 1\%$  in our study which means we require that the estimated number of signal events be positive and that the likelihood ratio test statistic be larger than 5.4. We generate  $1000\tau$  events each for the pure background sample and 1000 events of the "data sample", that is we assume a ratio of  $\tau$  between the sizes for the sideband and the signal region. We use  $\tau = 1/2, 1, 2$  and 4. Because we are testing the null there is no signal present. We find the likelihood ratio statistic using the three background estimates described above, except for model 6 where we include only the exact parametrization and the semiparametric fit. Here the signal location and width are fixed so there is no issue of the look-elsewhere

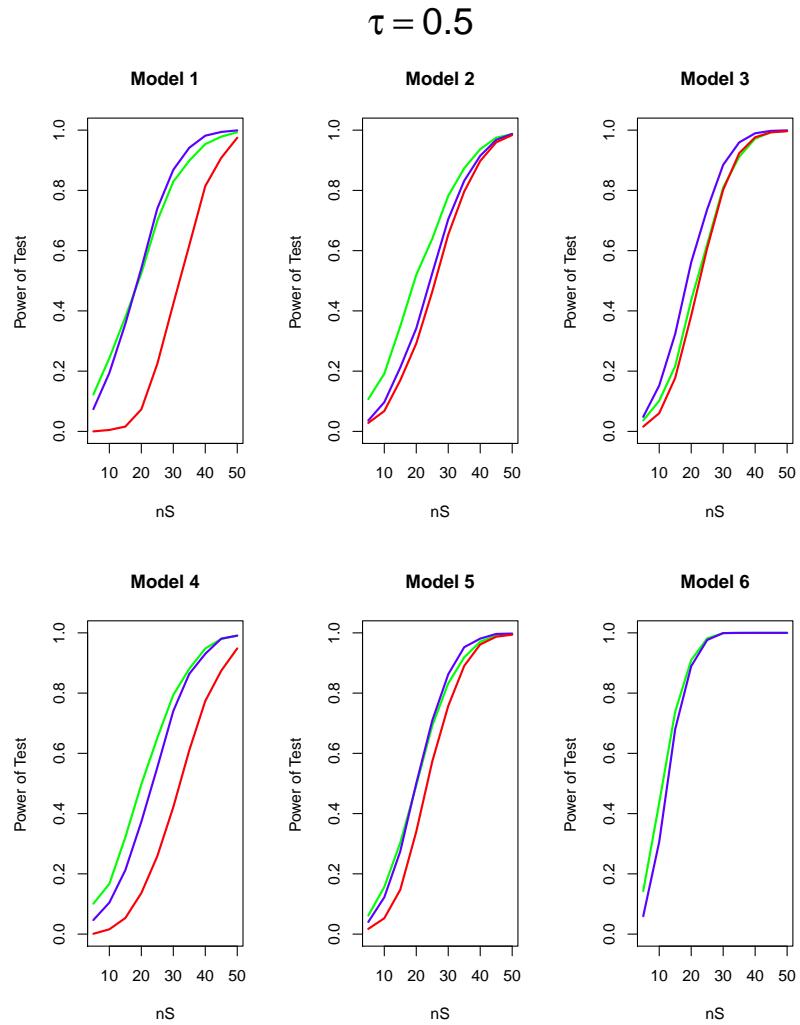
effect. This is repeated 25000 times. The curve for the exact parametrization is in blue, for the semiparametric method in green and for the false parametrization in red. We find



We see that using the exact parametrization the true type I error probability is just about the nominal one. Using the semiparametric method the true type I error probability is sometimes a little higher than the nominal one but approaches it for larger background samples. Using the slightly false parametrization it can be as much as 10 times the nominal one.

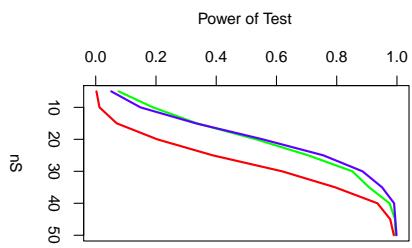
Next we add a signal drawn from a Gaussian distribution  $\mu = 0.35$  and  $\sigma = 0.02$ . Both data sets have 1000 events. We vary  $\alpha$  from 0.5% to 5%, equivalent to 5 to 50 signal events. Each simulation run is repeated 2500 times.

First, what is the power of the test, that is its ability to detect the signal?

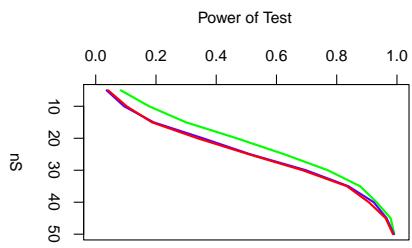


$\tau = 1$

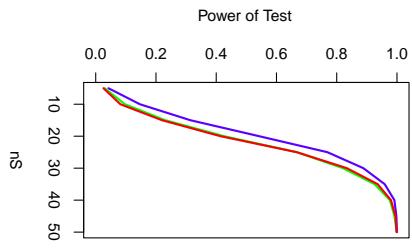
Model 1



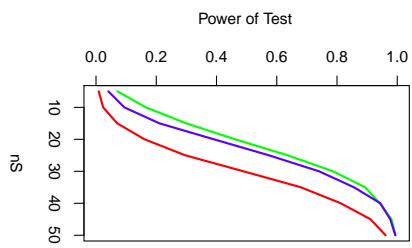
Model 2



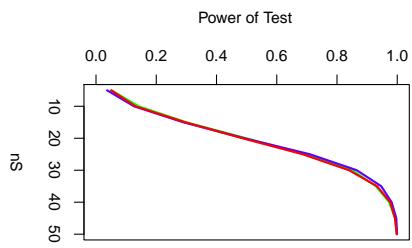
Model 3



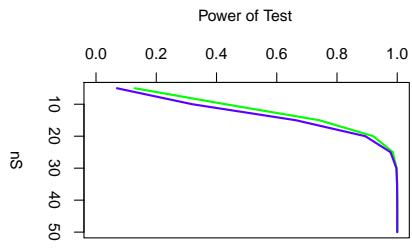
Model 4



Model 5

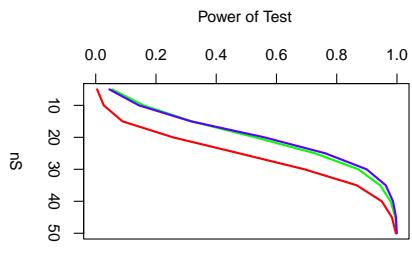


Model 6

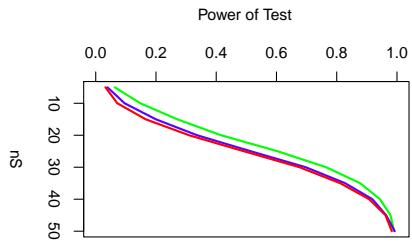


$\tau = 2$

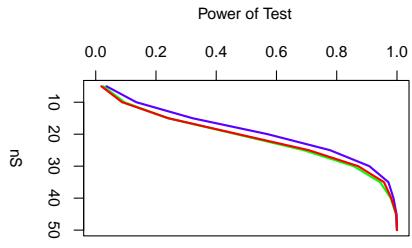
Model 1



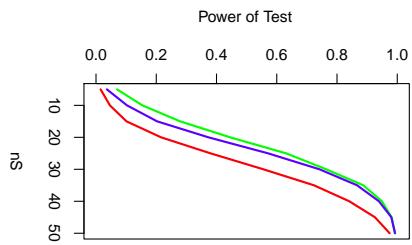
Model 2



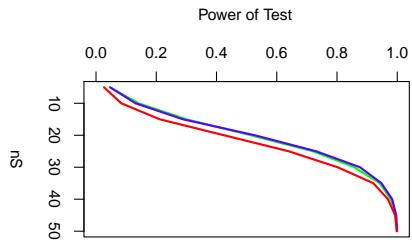
Model 3



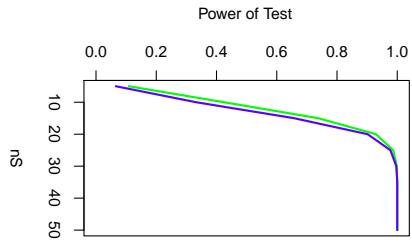
Model 4



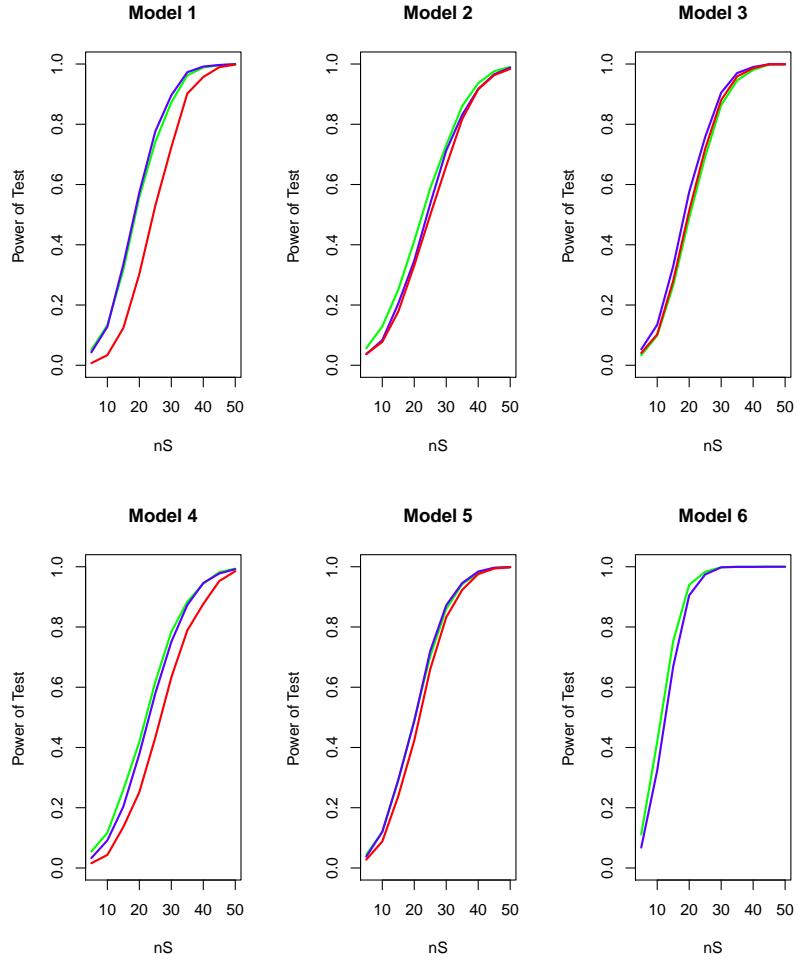
Model 5



Model 6



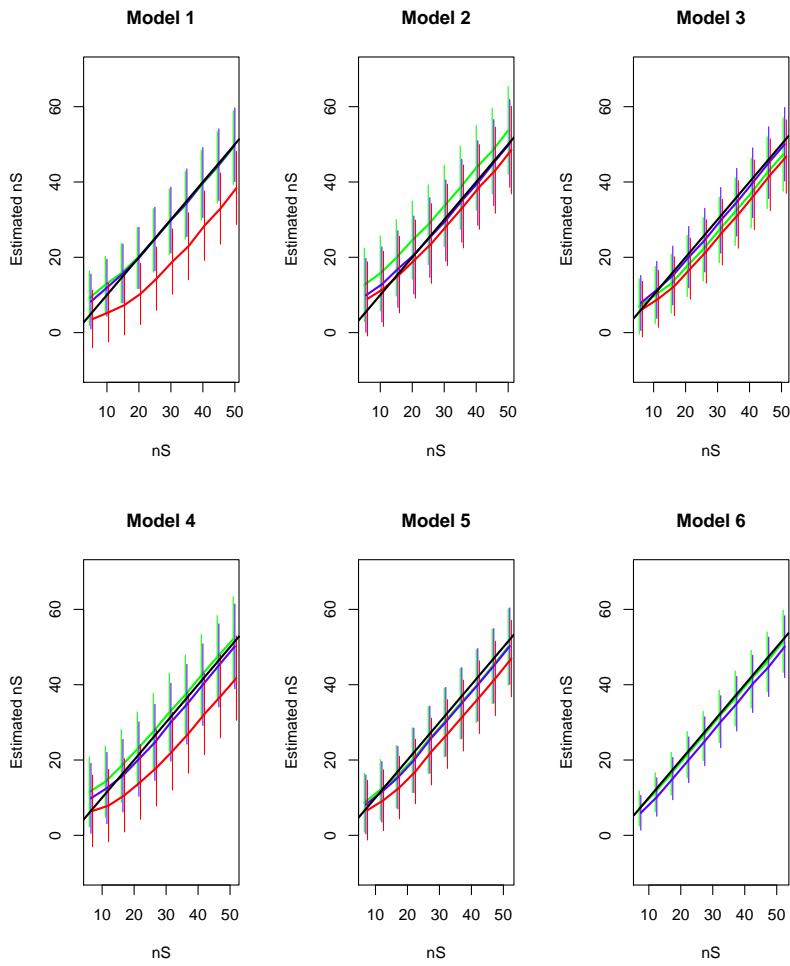
$\tau = 4$



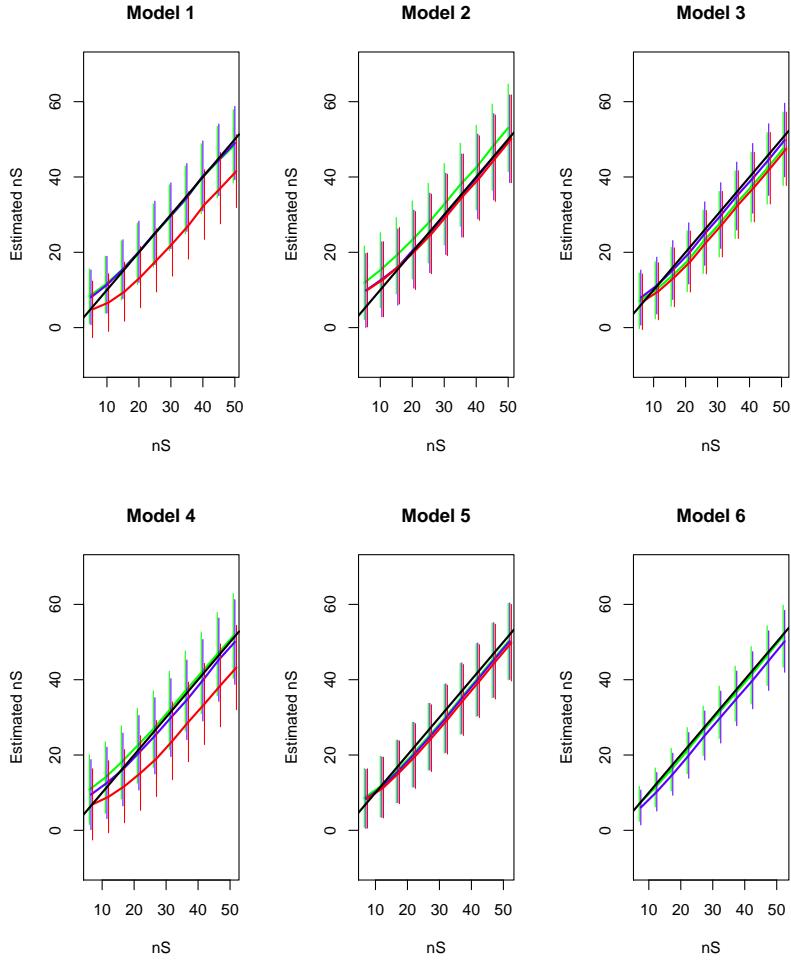
The power of the test based on the semiparametric fit is almost as good as the true parametric fit. Note that the power of the "false" parametrization is misleading because it already had a type I error that was too large.

Are they estimating the signal sizes correctly? Here are the means of the MC estimates:

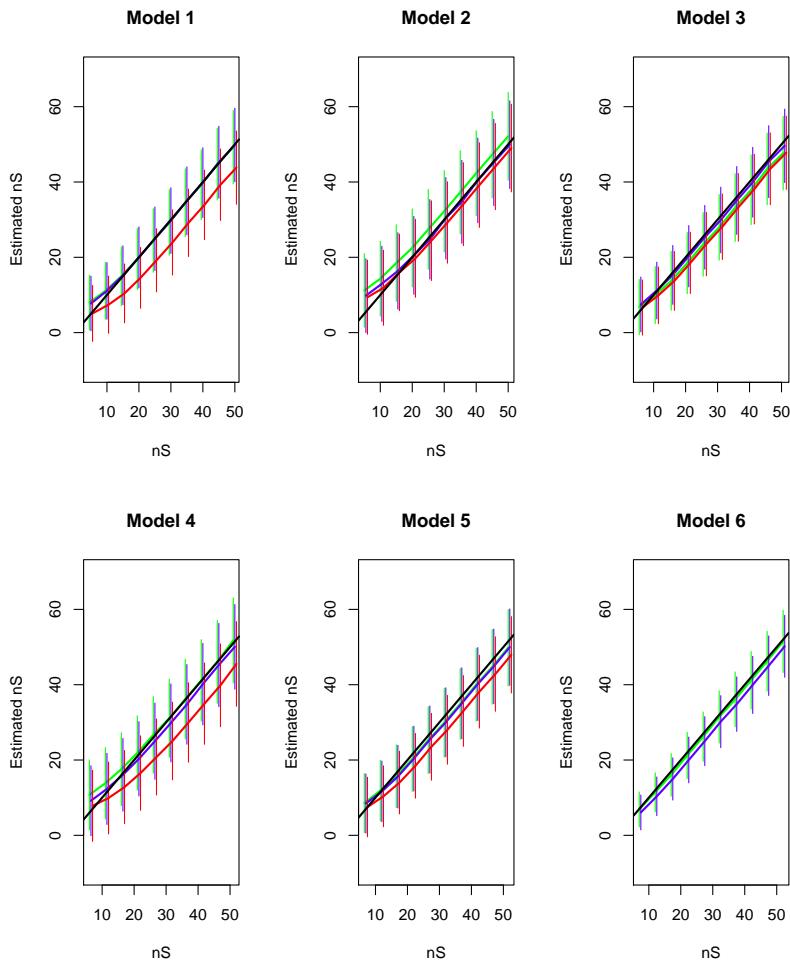
$\tau = 0.5$



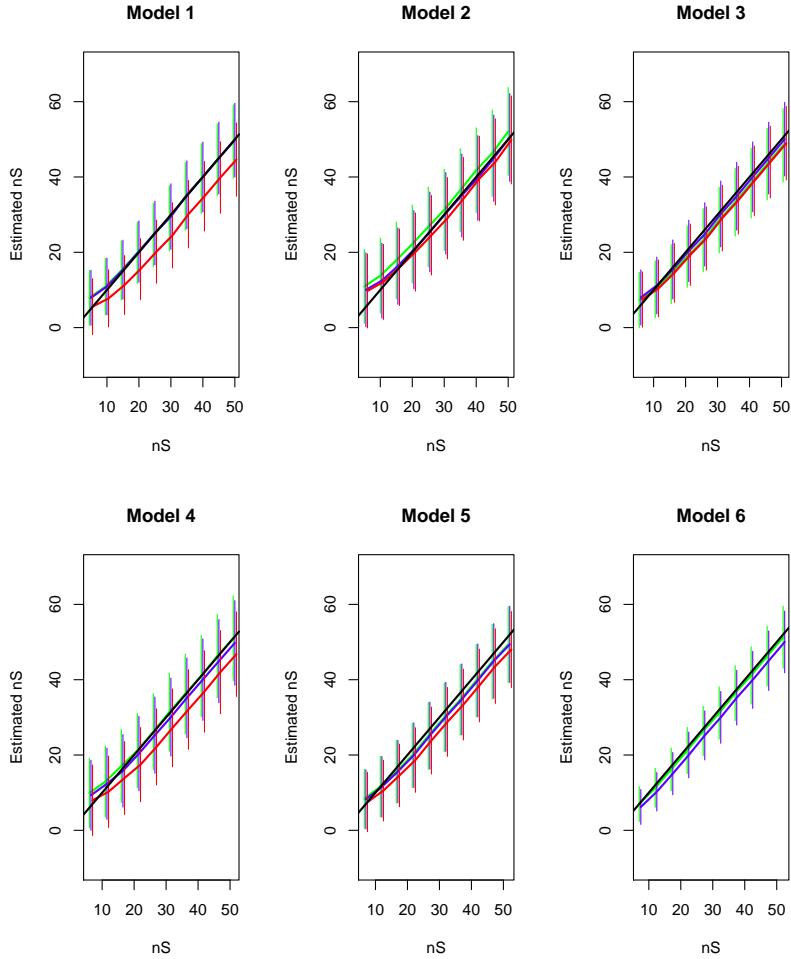
$\tau = 1$



$\tau = 2$



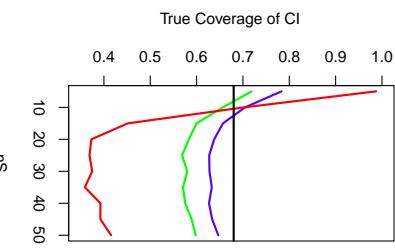
$\tau = 4$



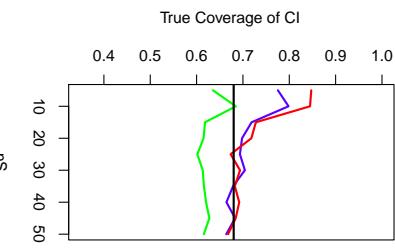
Are the error estimates correct? For this we will calculate the 1 standard deviation confidence intervals and check whether the percentage of intervals containing the true number of signal events is the required 68%:

$\tau = 0.5$

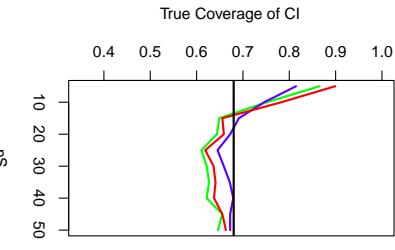
**Model 1**



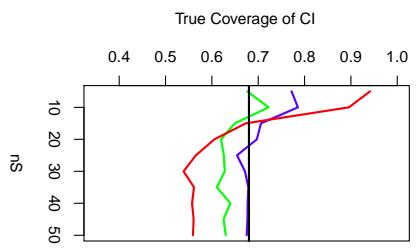
**Model 2**



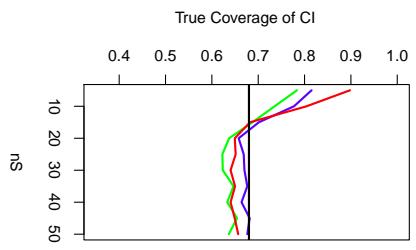
**Model 3**



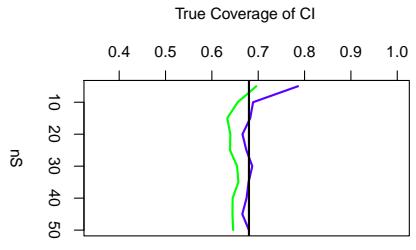
**Model 4**



**Model 5**

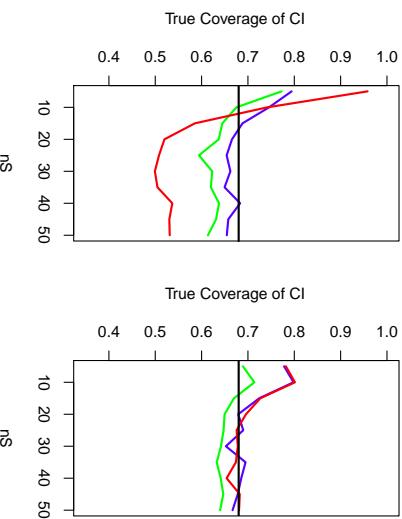


**Model 6**

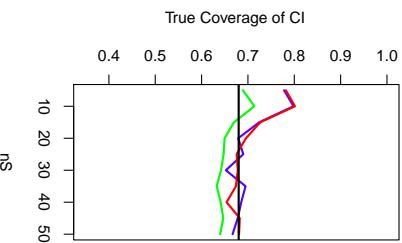


$\tau = 1$

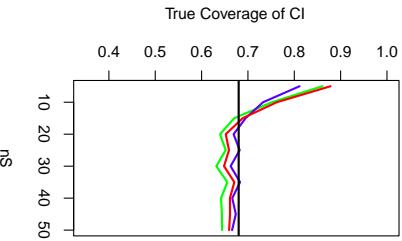
**Model 1**



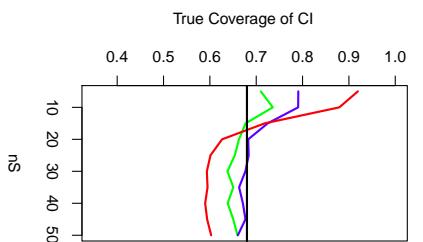
**Model 2**



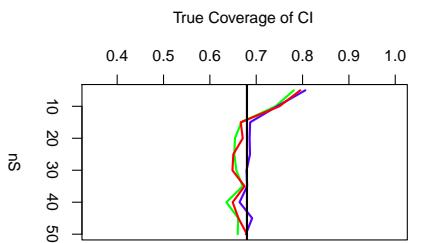
**Model 3**



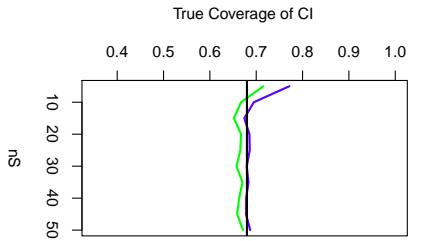
**Model 4**



**Model 5**

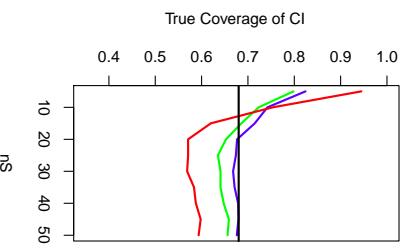


**Model 6**

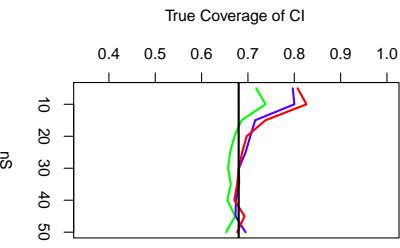


$\tau = 2$

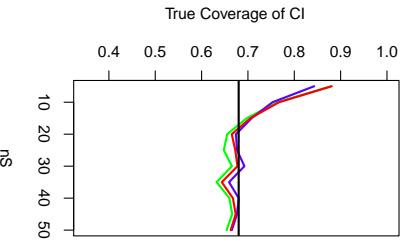
**Model 1**



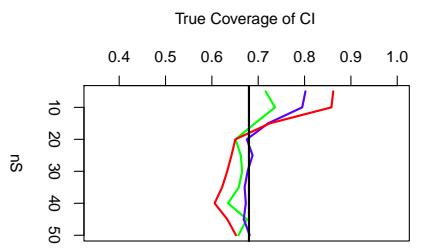
**Model 2**



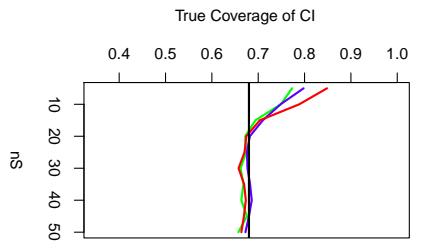
**Model 3**



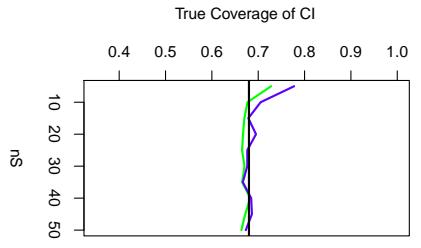
**Model 4**



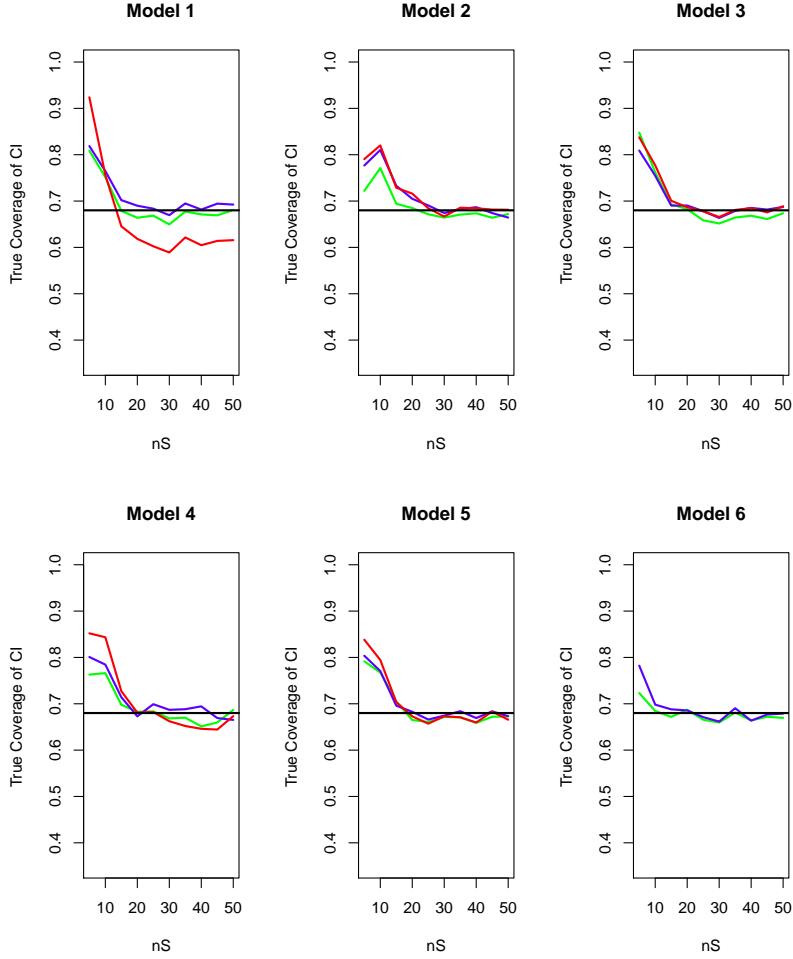
**Model 5**



**Model 6**



$$\tau = 4$$



Again the semiparametric method has errors just about correct, whereas the false parametrization can yield errors quite wrong.

Generally the semiparametric method does slightly worse than the exact parametrization (which unfortunately in real life is unknown) but better than a slightly false but "reasonably" looking one.

## 7. Computational Issues

The work described in this paper was done using the statistics package R, available at <http://cran.r-project.org/>, which includes the routine *density* for finding the optimal bandwidth  $h$  (with a choice of 5 different methods).

This routine was also used to calculate the density estimates. R uses function calls to C++ routines which are freely available. The HEP analysis platform RooFit, available at <http://roofit.sourceforge.net/>, includes the KEYS program written by Kyle Cranmer. In Perl, an implementation can be found in the Statistics-KernelEstimation module. In Java, the Weka package, available at <http://www.cs.waikato.ac.nz/ml/weka/>, provides weka.estimators.KernelEstimator, among others. In Gnuplot, kernel density estimation is implemented by the smooth kdensity option; the data file can contain weight and bandwidth for each point, or the bandwidth can be set automatically. C++ code for calculating the optimal bandwidth is available at <http://www.umiacs.umd.edu/labs/cvl/pirl/vikas> and is described in Raykar [47]. Recently developed algorithms for the so-called Fast Gauss Transform have made it possible to calculate the NPD estimates even for very large data sets. One example is discussed in Yang et al. [48]. A sample C routine which calculates limits for Model 2 using the semiparametric and the correct parametric background is available from the authors at <http://charma.uprm.edu/~rolke/publications.htm>.

## 8. Conclusions

We describe a method that uses nonparametric density estimation to estimate the background density and then finds estimates and errors for the parameters of the signal such as the number of signal events via maximum-likelihood. This is made feasible by the availability of a pure background sample. A simulation study shows that this method is quite competitive, almost as good as using the (in real life unknown) true parametrization and superior to using an almost correct parametrization. Moreover, because the nonparametric density estimate depends smoothly on the bandwidth, it is easy to do a sensitivity study and gain insight into the systematic uncertainty caused by different background shapes.

## 9. Acknowledgement

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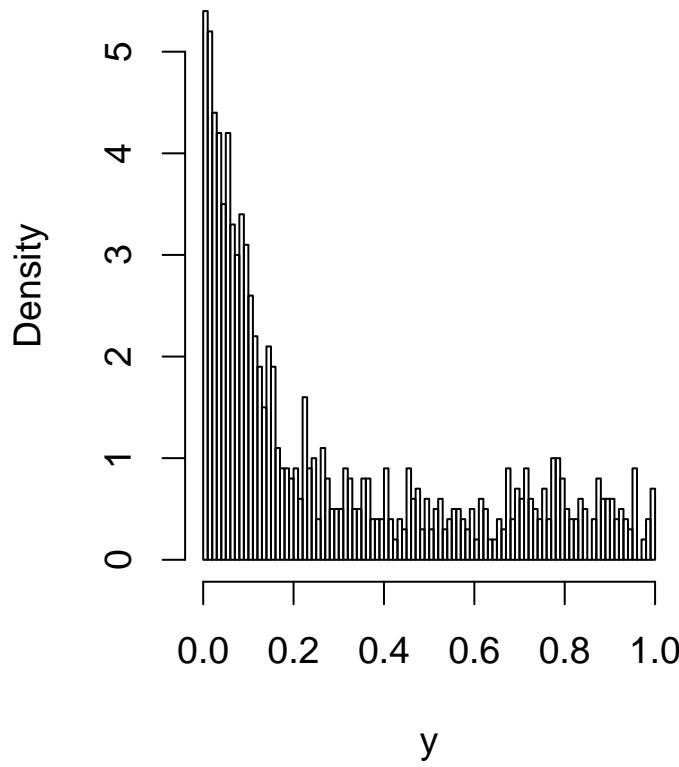
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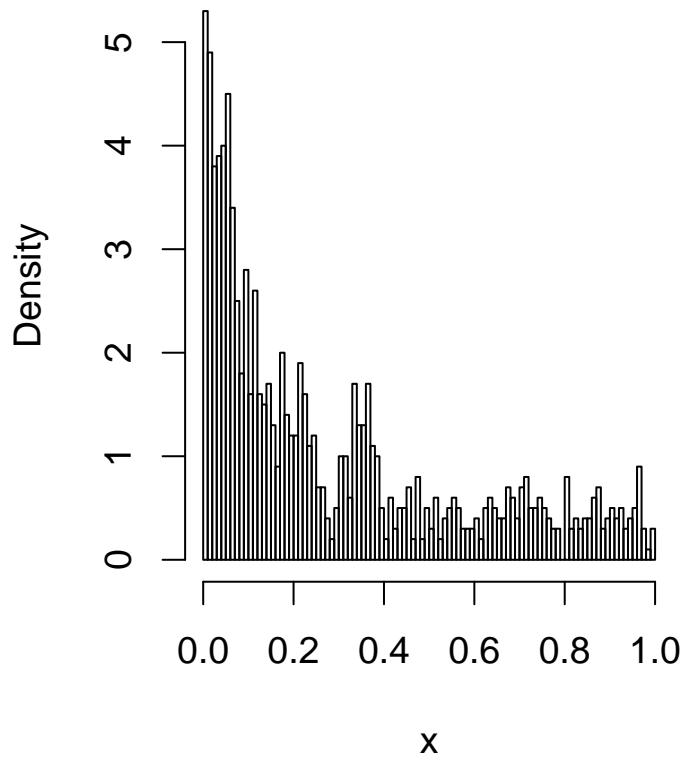
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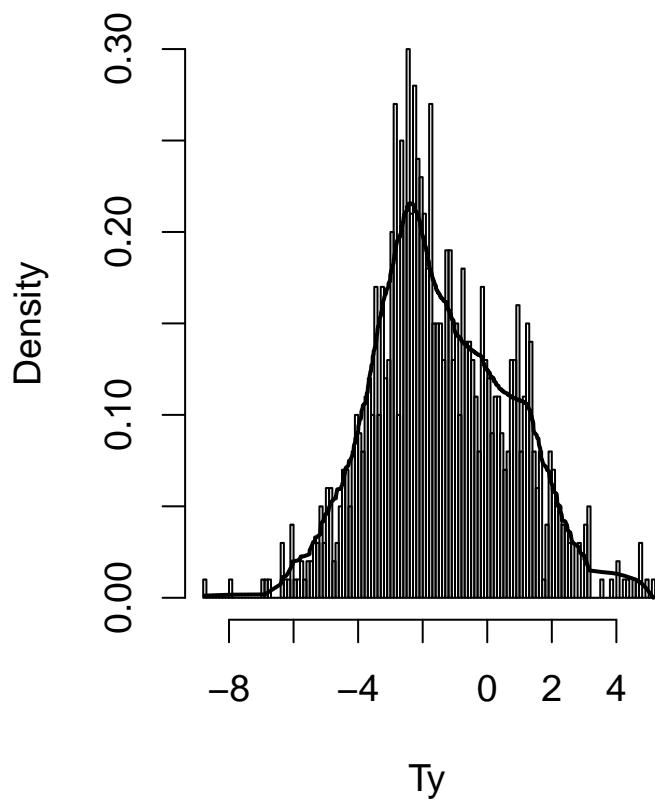
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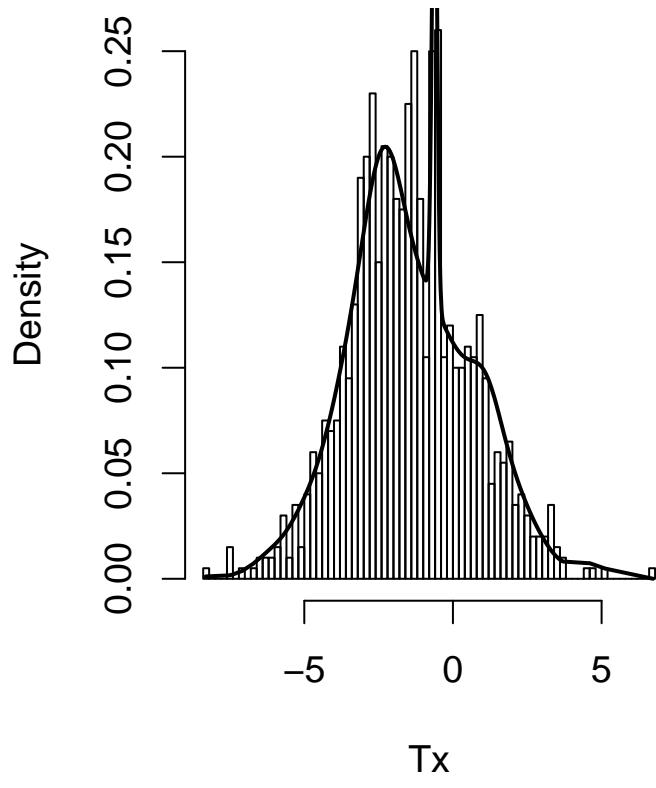
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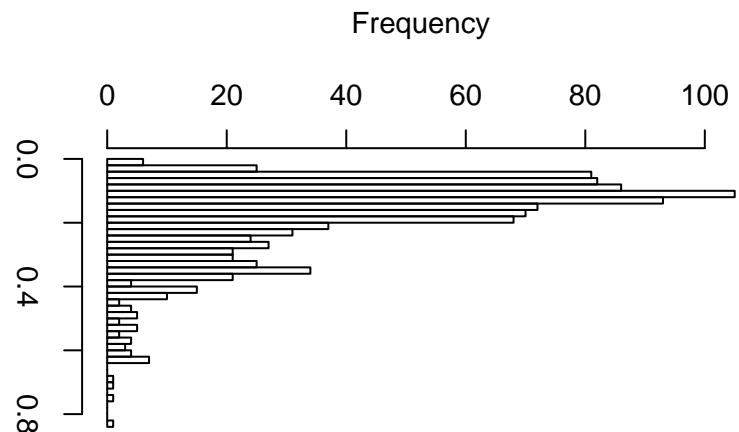
**Pure Background Sample  
Transformed**



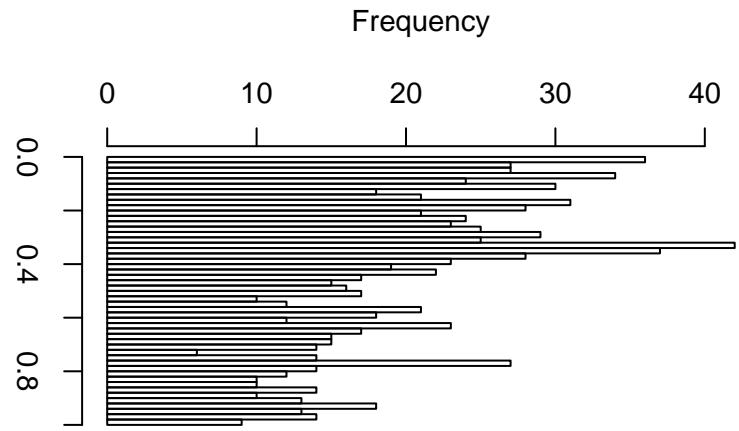
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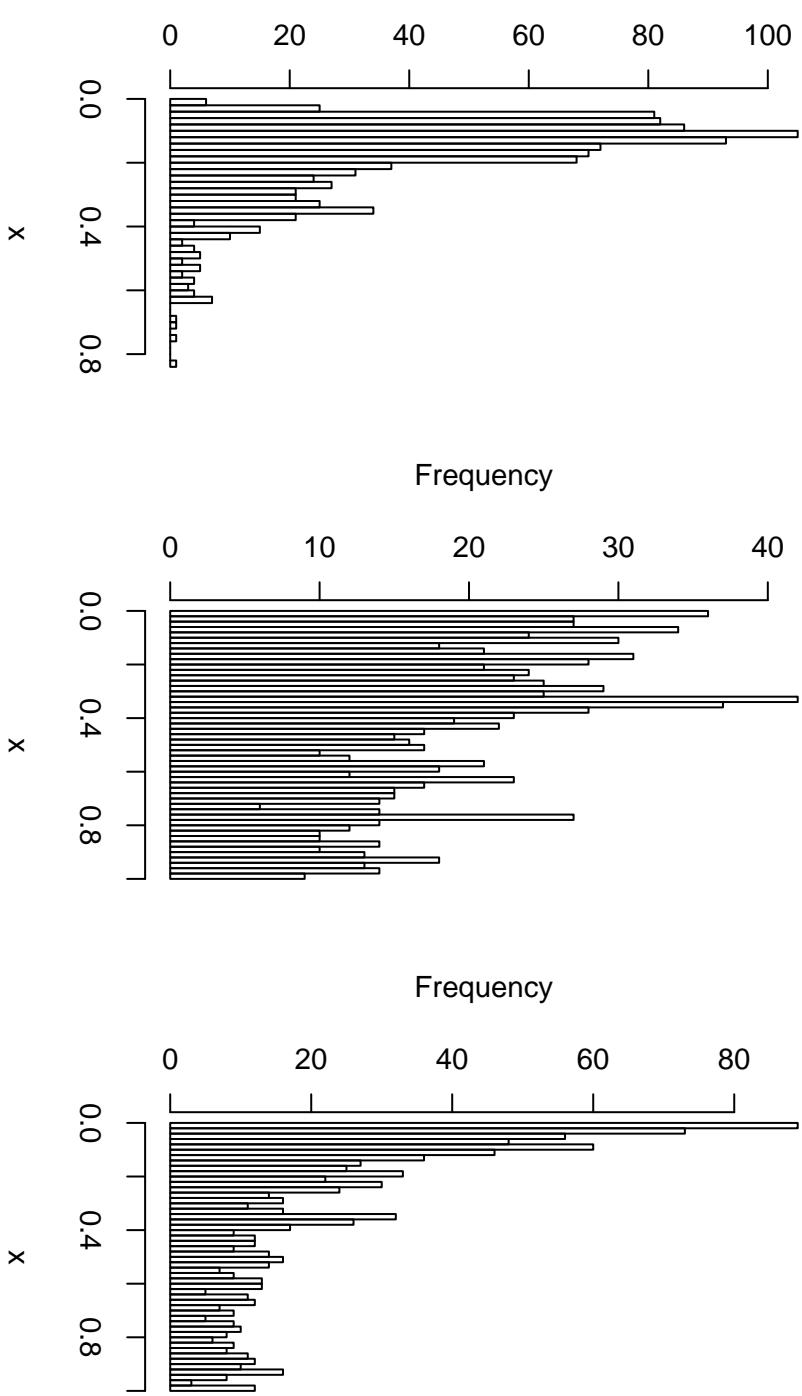
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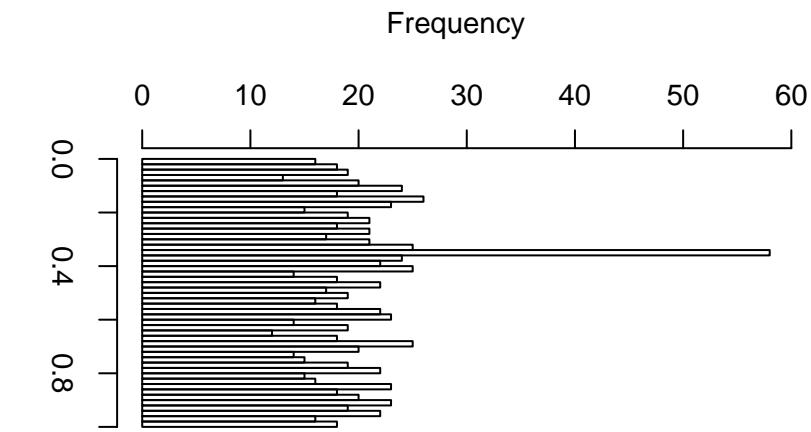
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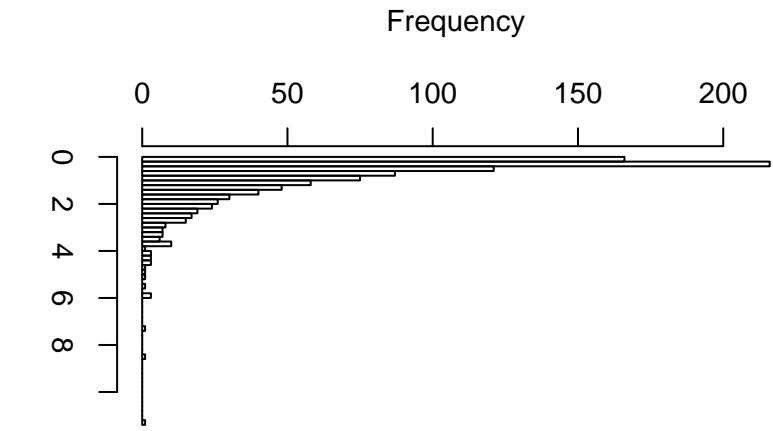
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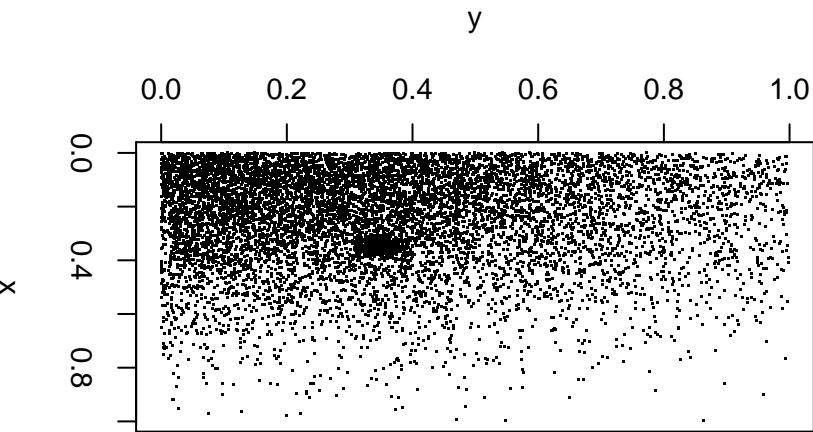
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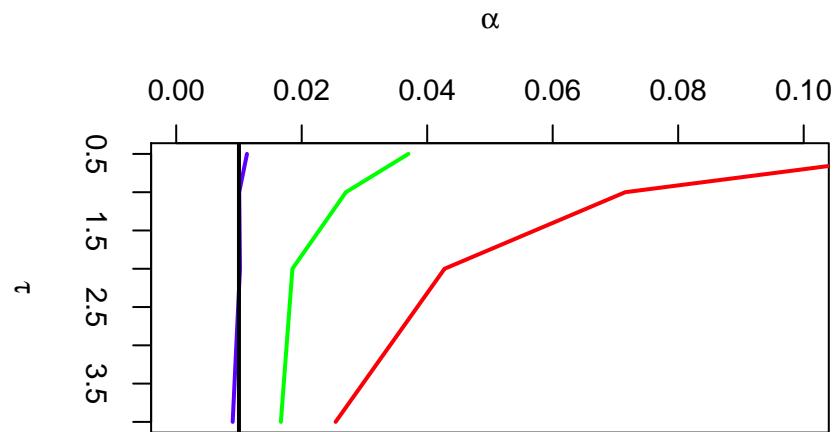


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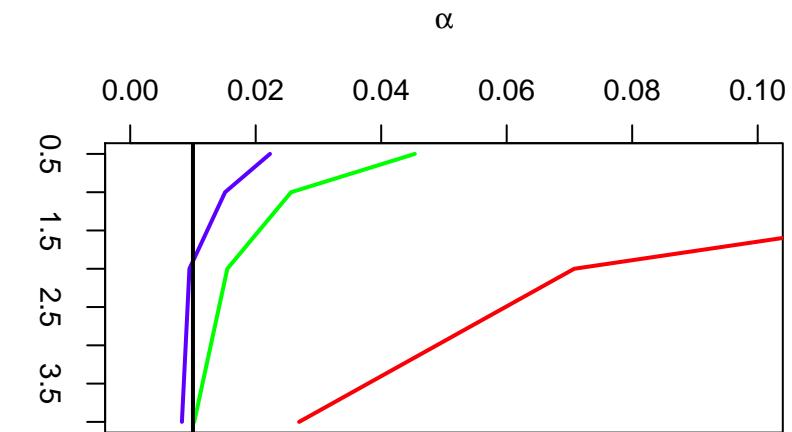


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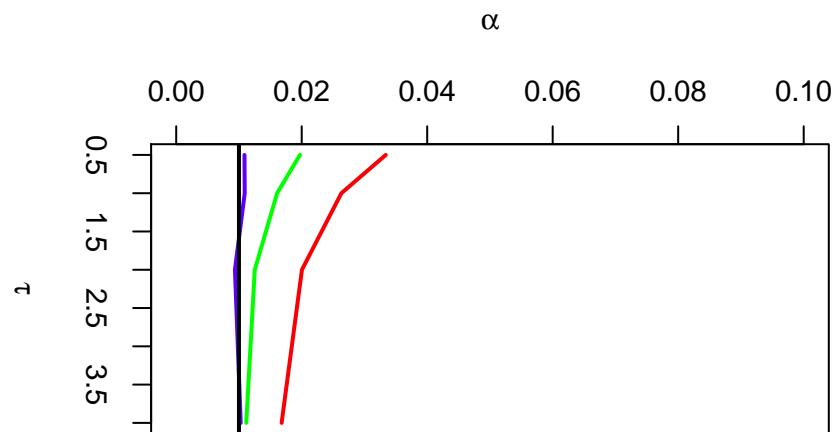




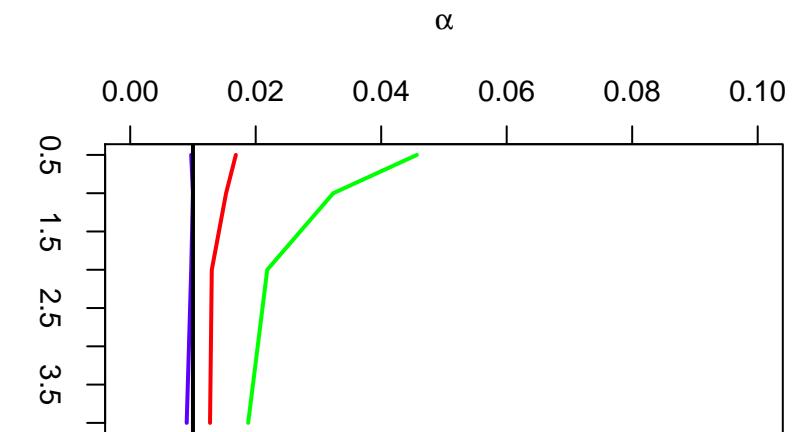
## Model 4



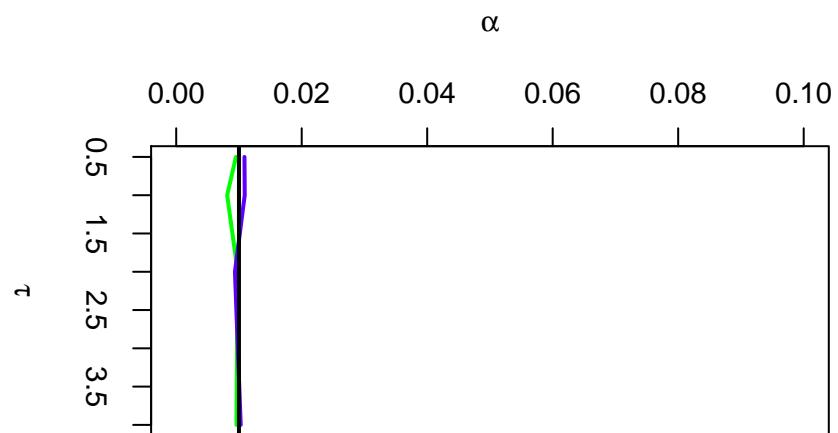
Model 1



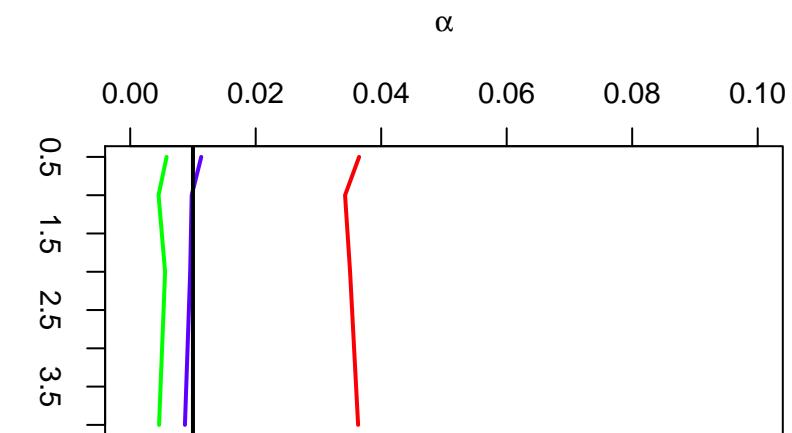
Model 5



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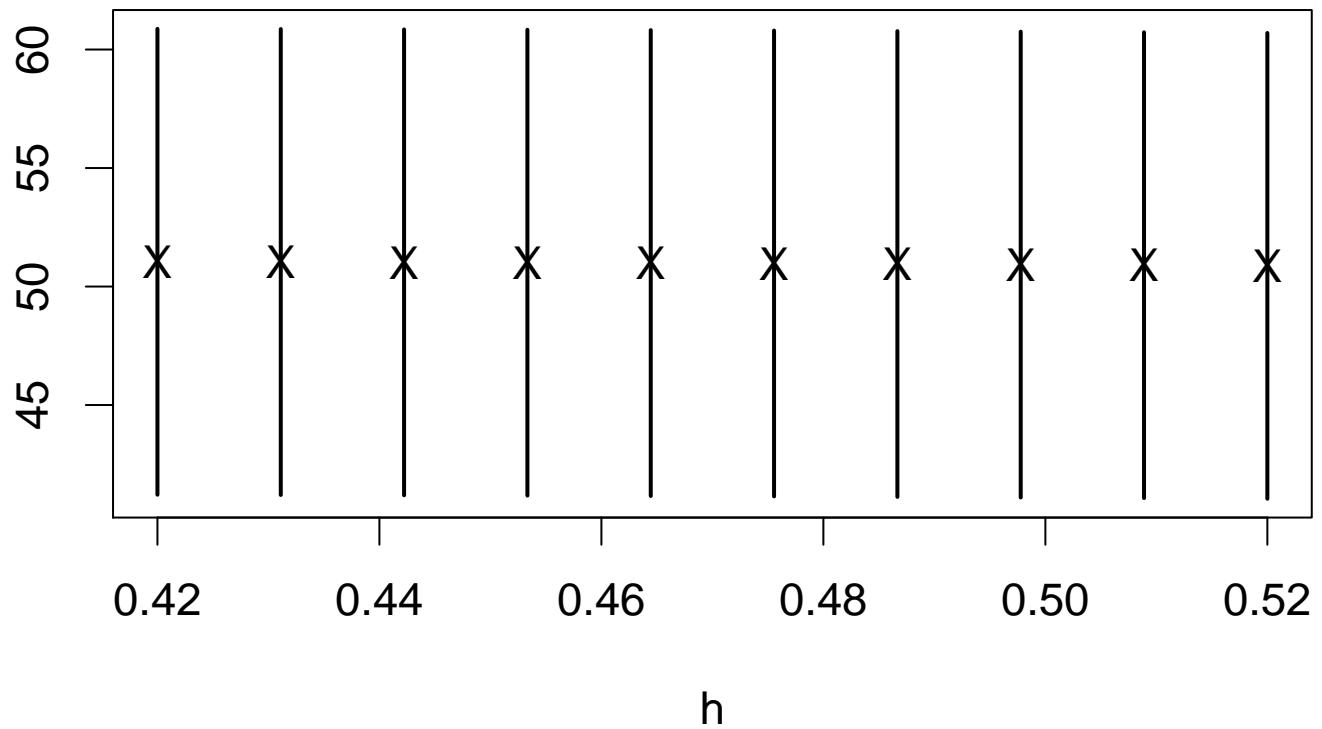


Model 6

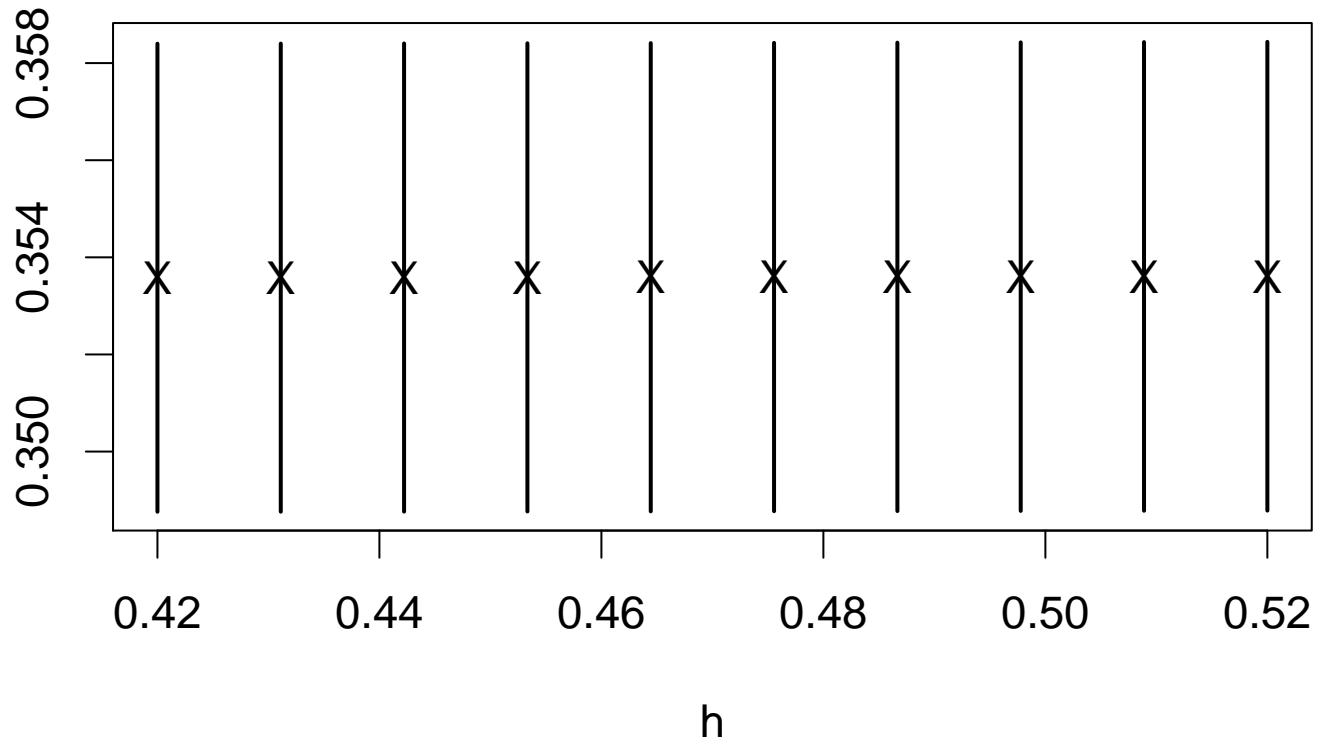


Model 3

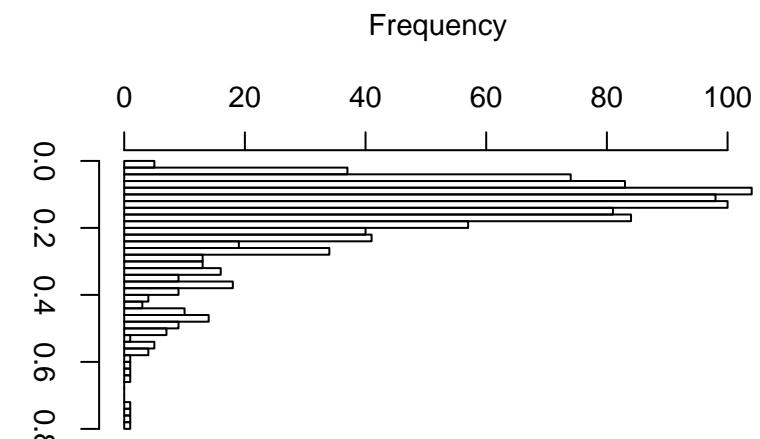
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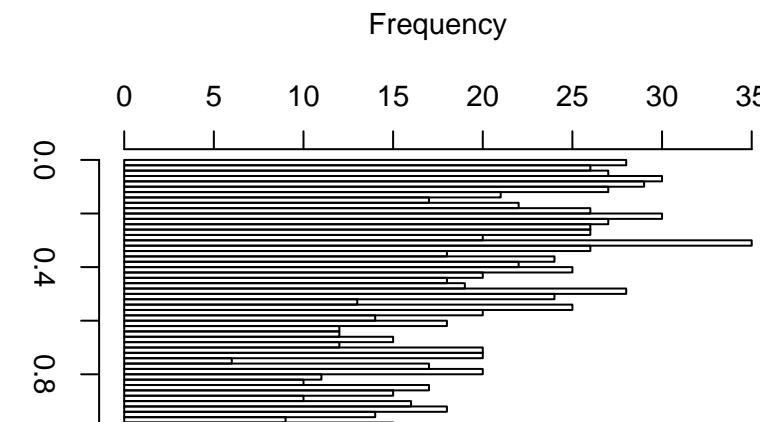
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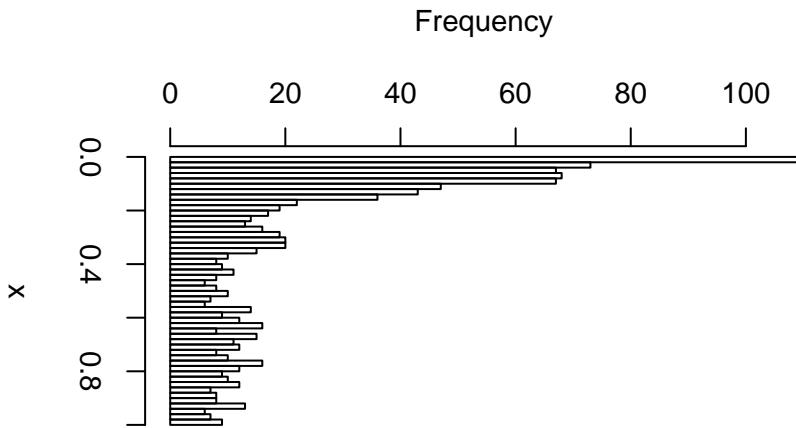
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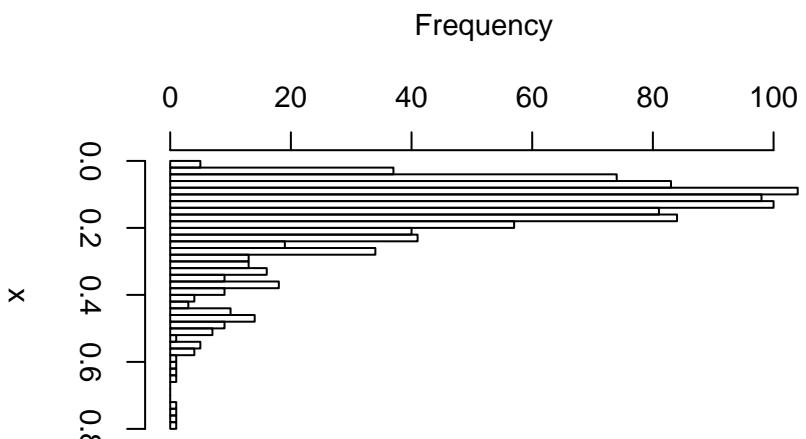
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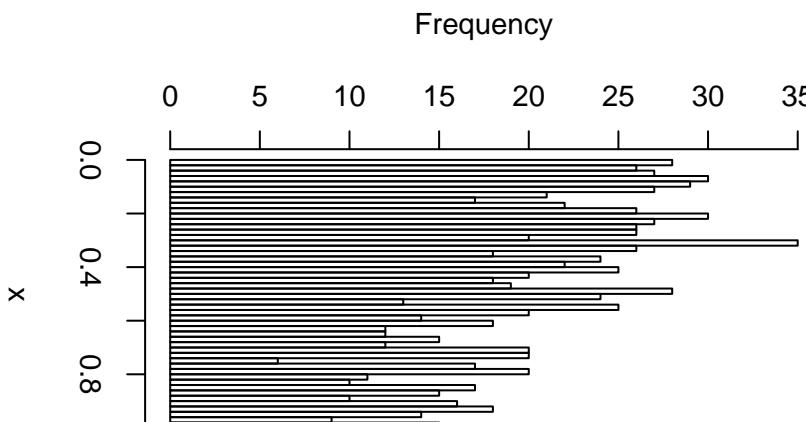
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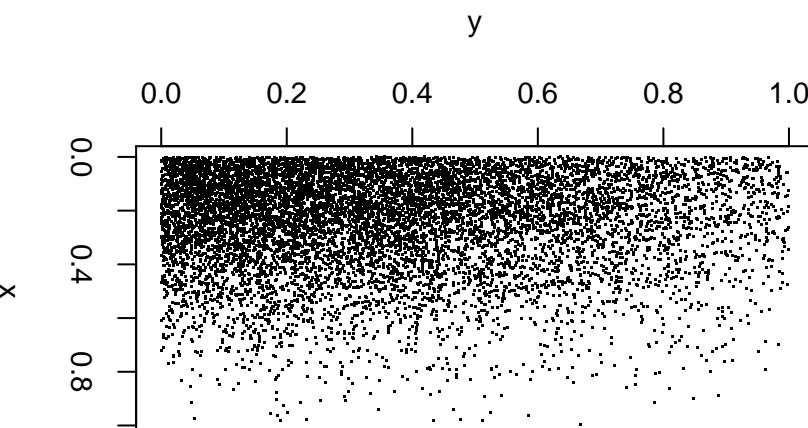
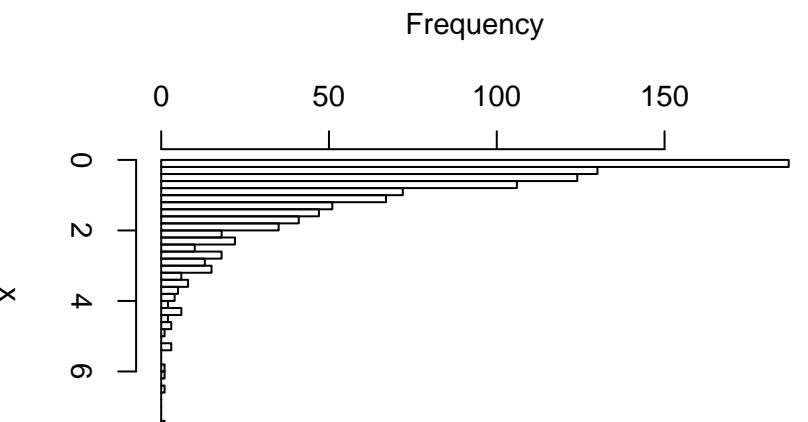
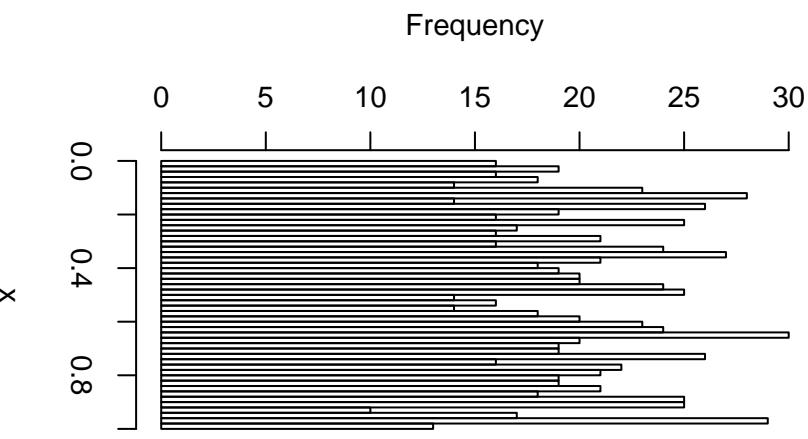
**Model= 4**



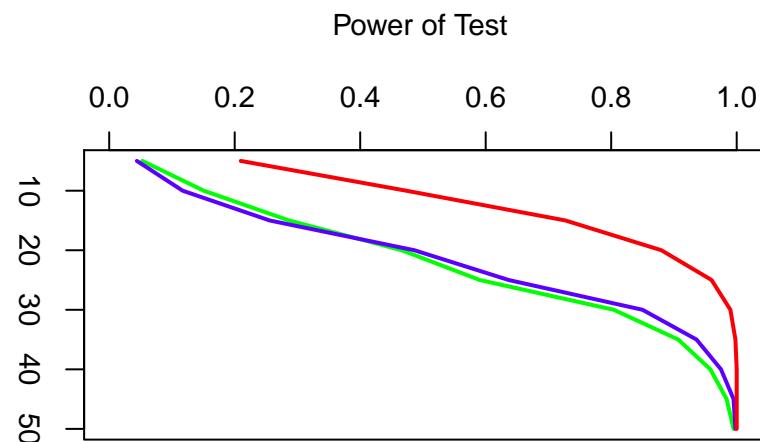
**Model= 5**



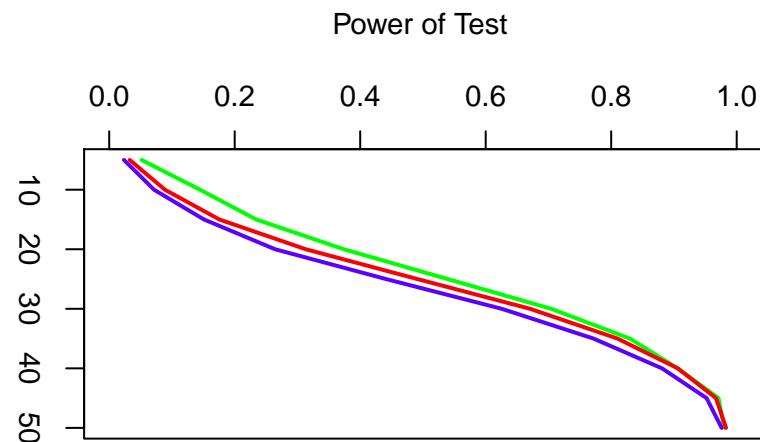
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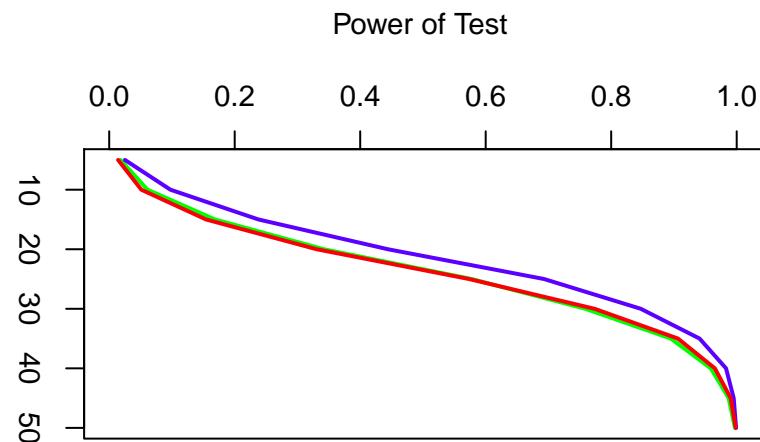
**Model 1**



**Model 2**



**Model 3**



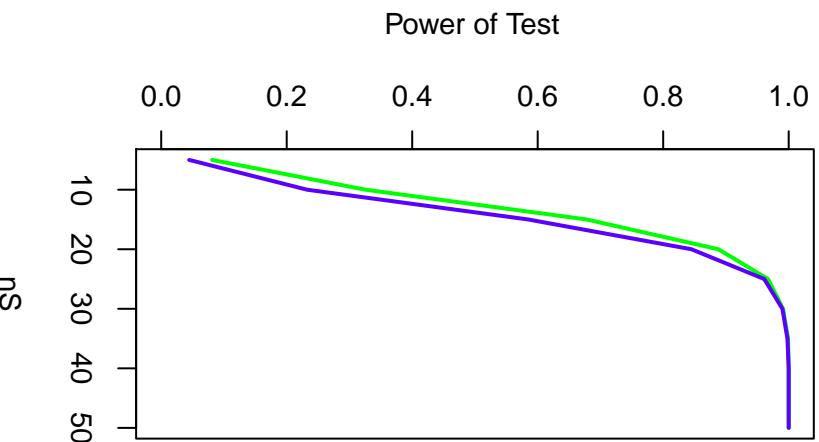
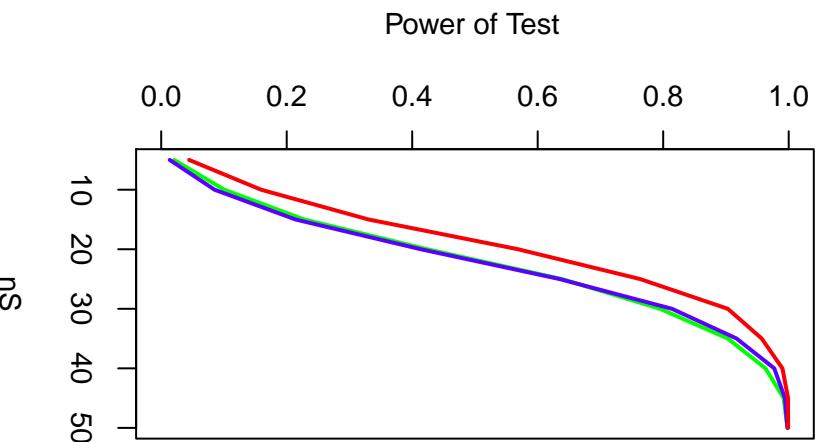
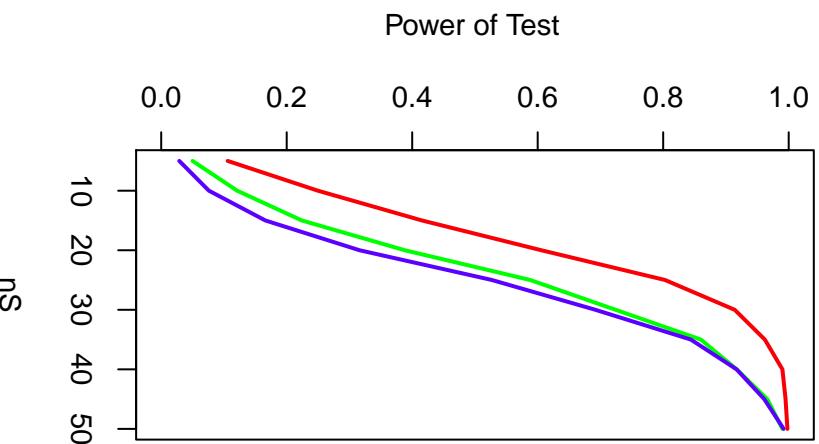
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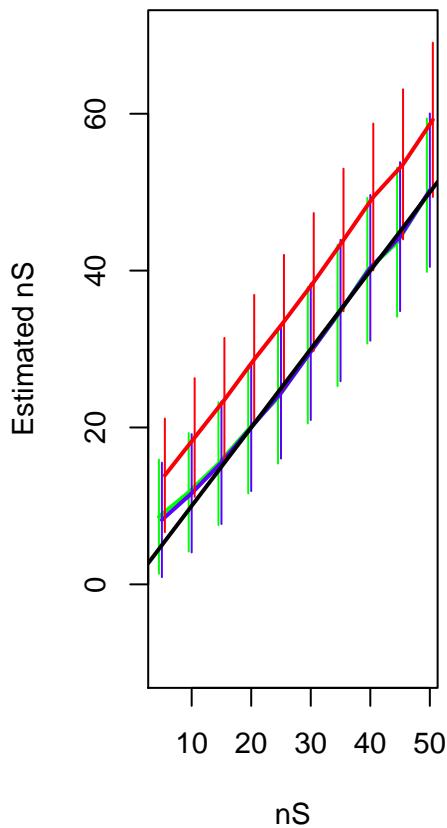
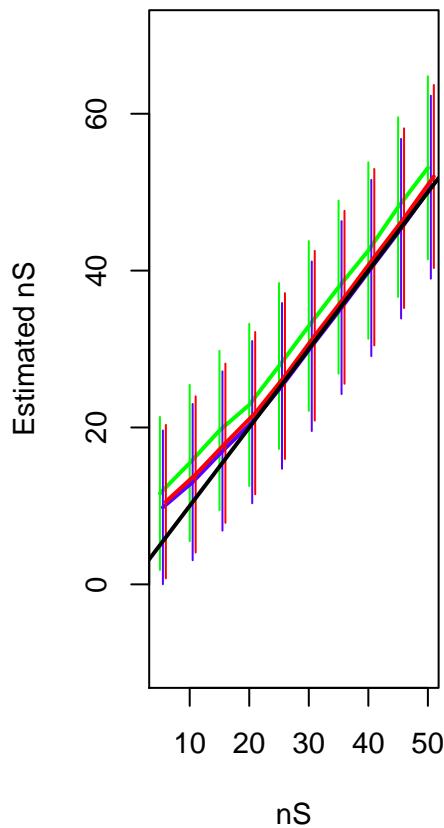
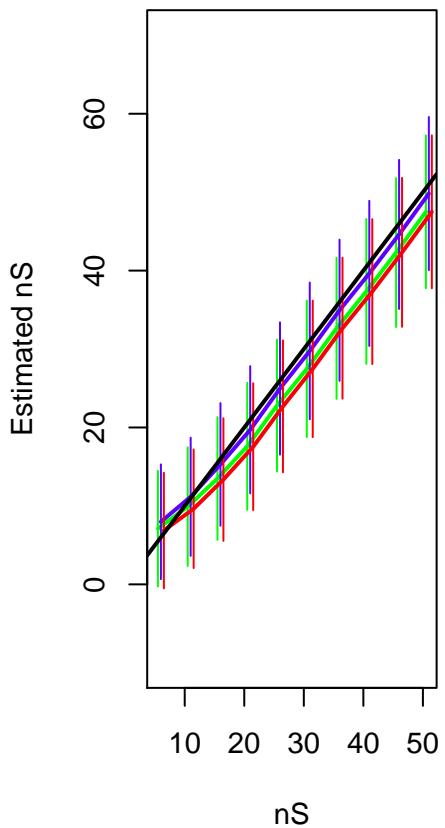
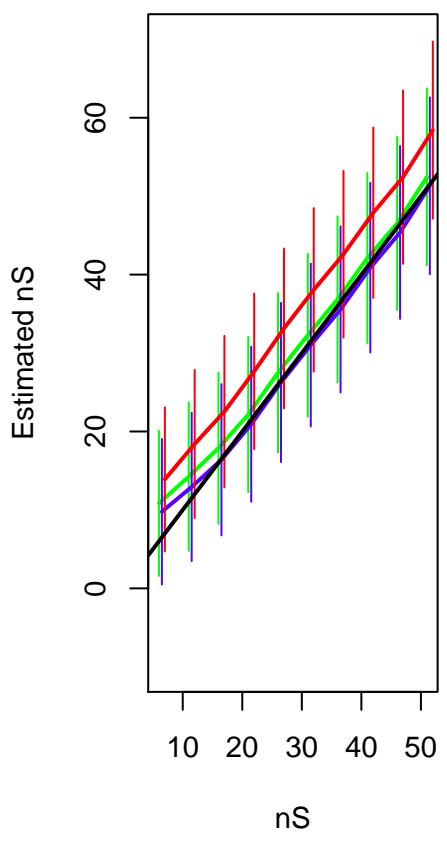
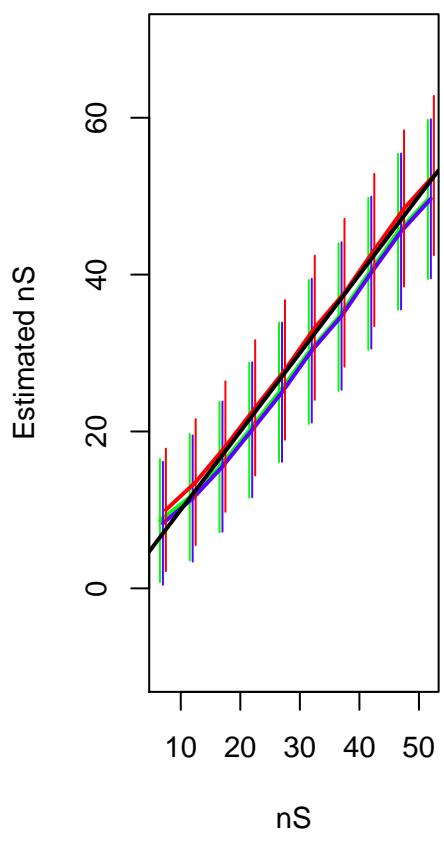
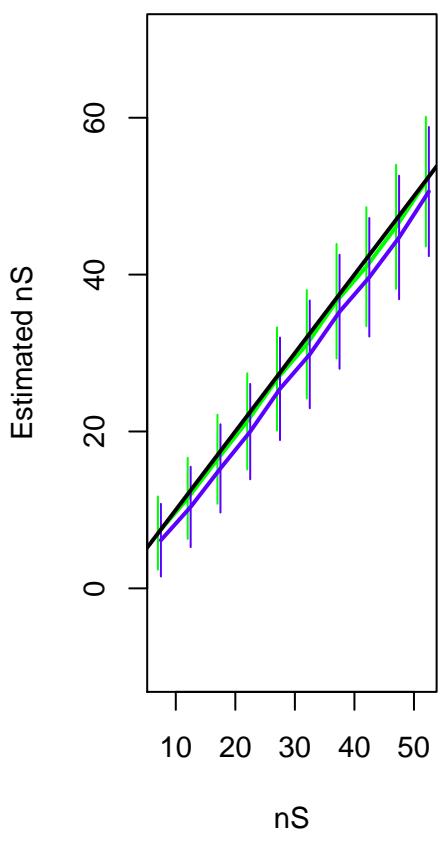


**Model 5**

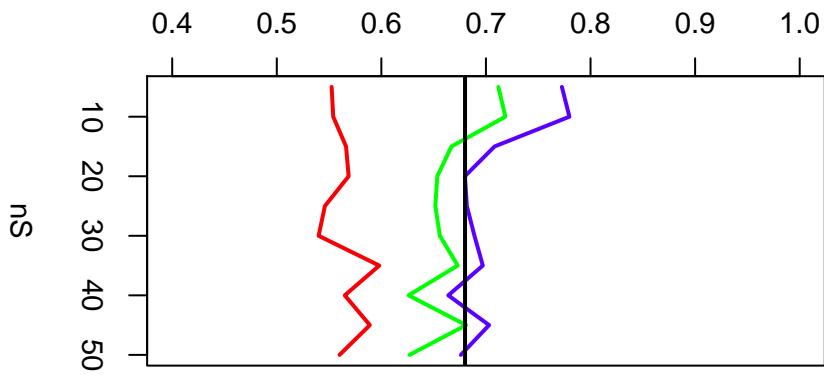


**Model 6**



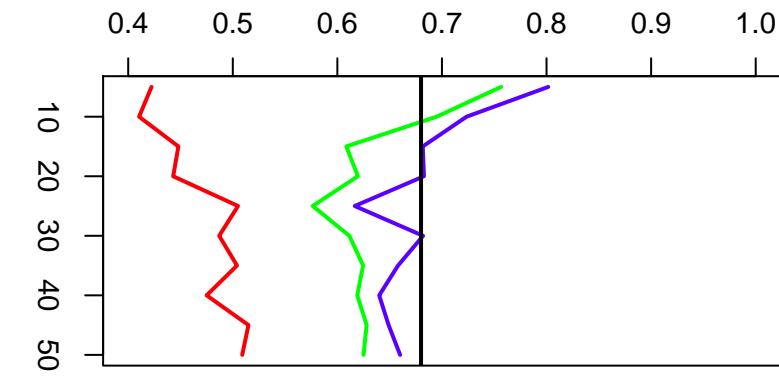
**Model 1****Model 2****Model 3****Model 4****Model 5****Model 6**

True Coverage of CI



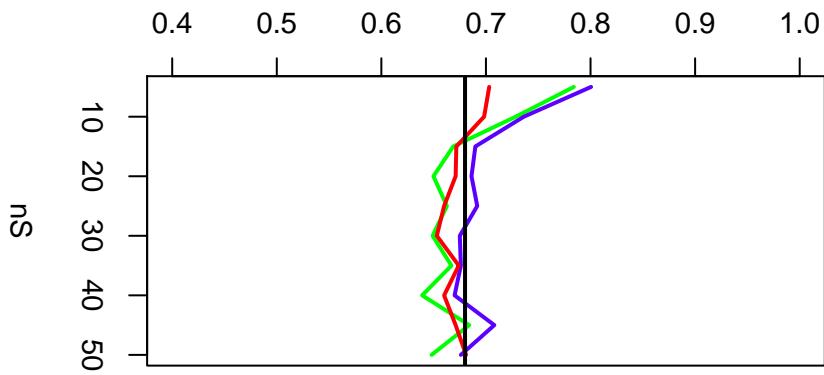
Model 4

True Coverage of CI



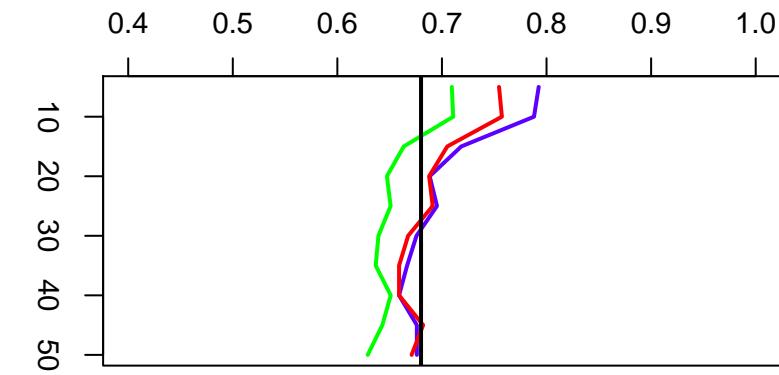
Model 1

True Coverage of CI



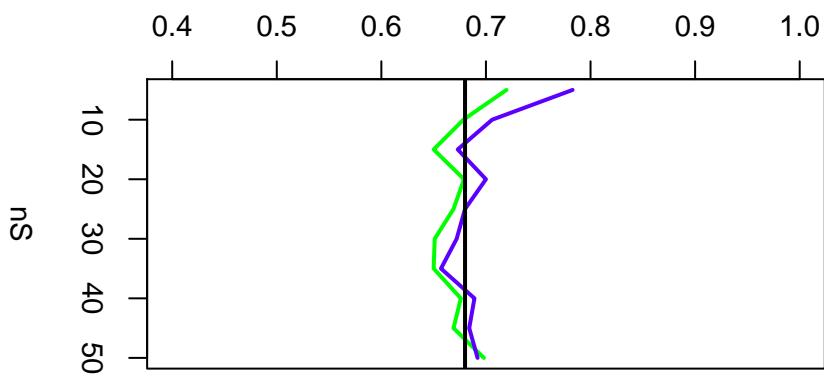
Model 5

True Coverage of CI



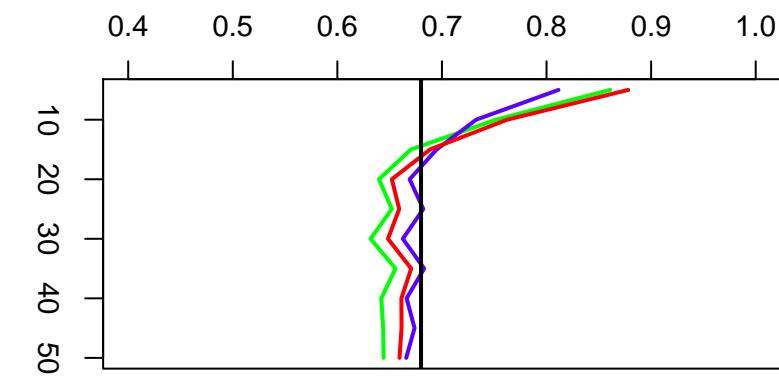
Model 2

True Coverage of CI



Model 6

True Coverage of CI



Model 3